Introduction to ordinary differential equations[1]

Let's consider the following problem.

A tank contains 50 *liters of some liquid* L_1 *and* 50 *liters of another liquid* L_2*. The liquid* L_2 *is added to the tank at the rate of* 5 *litres/min: at the same rate the solution is taken away. How much liquid* L_1 *will be in the tank after 1 hour?*

To arrive at the solution, the main difficulty is due to the fact that the percentage of liquid L_1 varies over time and to know how much of the liquid L_1 went out of the tank we must know the percentage of the liquid which is in the tank at the various instants.

the percentage of liquid L_1 varies over time

The idea which allows us to solve this problem is precisely to try to find how the percentage of the liquid L_1 varies along time. Starting point is the following *obvious* equation:

mass conservation → volume conservation

[volume of liquid L_1 present at some instant in time] = [volume of liquid L_1 present at some previous instant in time] - [volume of liquid L_1 went out in the interval between the two instants]

The problem is that the last term is not known. But, if we use this equation for a time interval "sufficiently small", we may reasonably assume that the percentage of liquid L_1 will not change during this interval. To reach *numerical* relations we still have to choose units of measure: for time we shall use minutes and for volume the liters. We shall denote by ϕ_1 (respectively: ϕ_2) how many liters of the liquid L_1 (respectively: L_2) are in the tank. Better, since these values change with time, we shall use the notations $\phi_1(t)$, $\phi_2(t)$: here, t represents how many minutes have passed from the instant "0", conventionally (and naturally) chosen as the instant at which the process starts.

the key idea!

choose the units of measure

Better

$\phi_1(t)$, $\phi_2(t)$

conventionally

(and naturally) chosen

So, we shall get, in a time interval Δt between the instants t_0 and $t_0 + \Delta t$, the following relation (*approximately* valid):

approximately

$$\phi_1(t_0+\Delta t) \text{ litres } = \phi_1(t_0) \text{ litres } -\Delta t \text{ min} \cdot \frac{\phi_1(t_0) \text{ litres}}{\phi_1(t_0) \text{ litres } + \phi_2(t_0) \text{ litres}} \cdot 5 \frac{\text{litres}}{\text{min}}$$

We stress that we have assumed that in all of the interval from t_0 to $t_0 + \Delta t$ the percentage of liquid L_1 is equal to the percentage of liquid L_1 present at the initial instant t_0.

Because $\phi_1(t_0)$ litres $+ \phi_2(t_0)$ litres $= 100$ litres , we get that:

trivial calculations

$$\phi_1(t_0 + \Delta t) \text{ litres } = \phi_1(t_0) \text{ litres } - (\frac{\Delta t}{20}) \cdot \phi_1(t_0) \text{ litres}$$

[1]The true title is: *Everything you always wanted to know about differential equations, but were afraid to ask.* But you are also liable: you did not you ask questions enough!

All of this holds as far as we refer to the volume of liquid in the tank. If we refer to the measures of these volumes, we get:

numbers only, eventually!

$$\phi_1(t_0 + \Delta t) = \phi_1(t_0) - (\frac{\Delta t}{20}) \cdot \phi_1(t_0)$$

That is:

et voilà!

$$\frac{\phi_1(t_0 + \Delta t) - \phi_1(t_0)}{\Delta t} = -\frac{\phi_1(t_0)}{20} \tag{1}$$

Notice that the left hand side is precisely a *difference quotient*, i.e. the average rate of change of ϕ_1 with respect to t.

WOW, the difference quotient!

At this point we can make the following *fundamental* remark. The relation (1) is only approximately true: the approximation, however, becomes better the more Δt becomes small. Mathematically, this amounts to pass to the limit:

a flash of genius: becomes better the more Δt becomes small

$$\lim_{\Delta \to 0} \frac{\phi_1(t_0 + \Delta t) - \phi_1(t_0)}{\Delta t} = -\frac{\phi_1(t_0)}{20}$$

That is: *if* this limit exists and is a real number, we get the "true" relation (I mean: not approximate any more):

WOW, the derivative!

$$\phi_1{}'(t_0) = -\frac{\phi_1(t_0)}{20} \tag{2}$$

What does it mean the "if" stressed above? It means that the relation (2) works to describe the evolution of the phenomenon *provided that we are a priori convinced* that the behavior of the phenomenon is regular: it means that ϕ_1 does not have discontinuities or even sudden variations (e.g. "corners" or "cusps" in the graph of ϕ_1).

Please be aware of the fact that this a "priori" regularity assumption has been made *before* getting the relation (1). Namely, if we have (1), we read immediately from it that the difference quotient (left hand side of this equation) is *constant*, since the right hand side does not depend on Δt, hence its limit will exist for sure! I mean, the a priori regularity assumption was made when we got the bright idea that on a "small" interval we can assume that the percentage of liquid L_1 is approximately constant. If we look more carefully at the assumptions we made, it is clear that we have assumed to be able to approximate "well" ϕ_1, on a small interval, with a linear function passing through $(t_0, \phi_1(t_0))$. Said otherwise, we assumed that ϕ_1 is differentiable.

the a priori regularity assumption is no joke!

Let us write:

$$\phi_1(t_0 + \Delta t) = \phi_1(t_0) - \left(\frac{\Delta t}{20}\right) \cdot \phi_1(t_0) + E(t_0, \Delta t)$$

If we want that it exists the limit of the difference quotient, we must *assume a priori* that

$$\lim_{\Delta t \to 0} \frac{E(t_0, \Delta t)}{\Delta t} = 0$$

Which means precisely that we assume that ϕ_1 is differentiable. Since ϕ_1 is a function of one real variable, being differentiable is equivalent to be derivable.

Let's go back to the body of the discourse.

The relation (2), clearly, holds for all $t_0 > 0$, and so:

holds for all $t_0 > 0$

$$\phi_1{}'(t) = -\frac{\phi_1(t)}{20} \quad \forall t > 0 \tag{3}$$

At this point we have to find the solution of (3), that is to find a function ϕ_1 satisfying the relation (3), and the answer to our problem will be $\phi_1(60)$. Or, to be precise, $\phi_1(60)$ liters.

A serious reader should try to find *by himself* a solution, before turning the page.

STOP!

Just not to leave empty all of this space, here are three good references on differential equations (in increasing order of appreciation on my side):

Braun, Martin: *Differential Equations and their Applications. An introduction to applied mathematics*, Springer, New York, 1978.

Pontryagin, Lev Semenovich: *Ordinary Differential Equations*, Addison Wesley, Reading (MA, USA), 1962.

Brauer, Fred e John A. Nohel: *Ordinary Differential Equations: a first course*, Benjamin, Reading (MA, USA), 1973.

A solution of (3) is immediately found: it is $\phi_1(t) = 0 \quad \forall t > 0$!!! Which means that the tank will never contain liquid L_1! How it can be? Where is the mistake?

OOOOH!

We did *forget* one thing. That (3) describes the *law of variation* of ϕ_1 as time varies: that things are really such is even more evident if we go back to the formula with Δt which was our starting point.

forget

now we see what is a differential equation: it describes a law of variation

We had:

back to the beginning, almost

$$\phi_1(t_0 + \Delta t) = \phi_1(t_0) - (\frac{\Delta t}{20}) \cdot \phi_1(t_0)$$

It is clear that this relation does not say *how much* is $\phi_1(t_0 + \Delta t)$. It just says how much is $\phi_1(t_0 + \Delta t)$ *if we already know* $\phi_1(t_0)$.

does not say *how much* is $\phi_1(t_0 + \Delta t)$

So, let us go back to the starting point, t =0. We have:

if we already know: HA!

$$\phi_1(\Delta t) = \phi_1(0) - (\frac{\Delta t}{20}) \cdot \phi_1(0)$$

If we know $\phi_1(0)$ we can then calculate $\phi_1(\Delta t)$. After this, we can calculate $\phi_1(2 \cdot \Delta t)$ by means of a similar relation, and so on. But we need to know $\phi_1(0)$. That is, we must know the "*initial conditions*". So, we can now be more precise: the correct formulation of the problem is not (3), but

The Cauchy problem

$$\begin{cases} \phi_1{}'(t) &= -\frac{\phi_1(t)}{20} \\ \phi_1(0) &= 50 \end{cases} \tag{4}$$

It is easy to verify that $\phi_1(t) = 50 \cdot exp(-t/20)$ solves (4), and so $\phi_1(60) = 50 \cdot exp(-3)$. So, after one hour the tank contains $50 \cdot exp(-3)$ liters.

A comment (by Fabrizio Rinaldi). It has been a big mistake, of course, not to "remember" the initial conditions. But in case we had done it consciously, not just because absent-minded, we could have been proud of the fact that we were looking for the law governing the "dynamics" (the evolution) of the problem *independently* of the specific initial conditions. Let's add, however, another word of warning: not all of the initial conditions make sense. For example, in our case, it does not seem reasonable to have an initial condition saying that there is a negative amount of liquid L_1 in the tank...

Now, we can say that we were able to find a good model which allowed us to solve the problem. But did we really solve it?

Another occasion for the serious reader to pause: think about it, before going to the next page.

STOP!

Not at all. We still have to ask for additional fundamental properties. The uniqueness of the solution of (4), just to begin with: if there were another solution $\tilde{\phi}_1$ besides the one we found, we could not be any more able to say how many liters are left at the end. It will be essential to be able to guarantee *existence*[2] and *uniqueness* of the solution of (4), if we pretend to say that we found a model which is adequate for our problem.

I say **NO**!
uniqueness
existence

Actually, there are additional requirements: the *continuous dependence of the solution upon the data*, for example. If, starting with $\phi_1(0) = 50.001$, the solution were quite different from the one we got for $\phi_1(0) = 50$, could we say that we are satisfied?

continuous dependence upon the data

The various theorems about existence, uniqueness and continuous dependence on the data have precisely the role of telling us whether and when we can be confident that these properties hold. A technical discussion of these results is out of the scope of this introduction: on these issues I refer to standard texts on differential equations.

theory is useful...

Let's come back, instead, to a point that we have overlooked. It seemed to be a good idea to work on a "small" interval Δt: actually, the smaller we take it, the better is the approximation provided by the formula (1). *But*: if we shrink Δt, the improvement in the approximation achieved on the single interval is not destroyed by the fact that we must consider *a greater number of intervals*? The answer is, luckily, no, and the reason is due to the fact that the error on the single interval is of a higher order with respect to the length of the interval (keep in mind the definition of differentiability!) and so we have an overall improvement taking smaller and smaller intervals. Let us see in details, with an example, what happens. To get the result I am interested in, I will make assumptions which are stronger that those actually needed, just to see more easily what happens: namely, I will assume that $\phi \in \mathcal{C}^2([0,T])$.

a *sane doubt* assails us
be honest: did you consider this?
these calculations are for those full of good will

Let us divide $[0,T]$ into n identical intervals: in the k-th interval, $k = 0, 1, \ldots, n-1$, we have ($t \in [kT/n, (k+1)T/n]$):

$$\hat{\phi}(t) = \hat{\phi}(kT/n) + \phi'(kT/n) \cdot (t - (kT/n)); \quad \hat{\phi}(0) = \phi(0)$$

It means, we substitute ϕ with the "piecewise linear" approximation $\hat{\phi}$ defined as follows:

$$\hat{\phi}(t) = \phi(0) + (T/n) \sum_{j=1}^{k-1} \phi'(jT/n) + \phi'(kT/n) \cdot (t - (kT/n)),$$

for $t \in [kT/n, (k+1)T/n]$ if $k = 0, 1, \ldots, n-1$

[2]Existence! Not all of the problems are easy as this one, and maybe we don't have the "magical formula" to solve it.

To keep things as simple as possible[3], let us just estimate $|\hat{\phi}(T) - \phi(T)|$. It is

$$|\phi(T/n) - \hat{\phi}(T/n)| = |\phi(T/n) - [\phi(0) + \phi'(0) \cdot (T/n)]| =$$

(Taylor's formula, with Lagrange's reminder) $= |\phi''(\xi_1) \cdot (T^2/n)| \leq (MT^2)/n^2$ (since $\phi \in \mathcal{C}^2([0,T])$, there exists $M \in \mathbb{R}$ s.t. $|\phi''(t)| \leq M$ for all $t \in [0,T]$).

Then, we get:

$$|\phi(2T/n) - \hat{\phi}(T/n)| = |\phi(2T/n) - [\hat{\phi}(T/n) + \phi'(T/n) \cdot (T/n)]| \leq$$

$$\leq |\phi(T/n) - \hat{\phi}(T/n)| + |\phi(2T/n) - [\phi(T/n) + \phi'(T/n) \cdot (T/n)]| \leq$$

$$\leq (MT^2)/n^2 + |\phi''(\xi_1) \cdot (T^2/n^2)| \leq 2(MT^2)/n^2$$

And so on. We get, after n steps, the estimate:

$$|\phi(T) - \hat{\phi}(T)| \leq n \cdot (MT^2)/n^2 = (MT^2)/n$$

Which means: it is convenient to take shorter intervals, because the approximation we get is better.

Let me notice that such kind of considerations lie at the root of the proof of an important existence theorem for (4). Let me further notice that what we did was, essentially, to approximate the solution via the so-called Euler method, which is a numerical method for solving differential equations[4].

One additional thought about the problem that we are considering. The model that we built to solve it brought us to a differential equation. We could have followed a different road (but with many parts in common) to reach an *integral equation*. Let us see how. The idea is that the amount of liquid L_1 in the tank at a given moment is given by the initial quantity of liquid, minus the amount of liquid that went out in all of the intervals preceding the moment that we are considering. Using, for sake of convenience, intervals of equal length $\Delta t = \frac{t}{n}$, we get: an (almost) NEW road!!!

$$\phi_1(t) = \phi_1(n\Delta t) =$$

$$= \phi_1(0) - (\Delta t/20) \cdot \phi_1(0) - (\Delta t/20) \cdot \phi_1(\Delta t) - \ldots - (\Delta t/20) \cdot \phi_1((n-1) \cdot \Delta t) =$$

$$= \phi_1(0) - \sum_{k=0}^{n-1} \frac{\phi_1(k\Delta t)}{20} \cdot \Delta t = \phi_1(0) - \sum_{k=0}^{n-1} \frac{\phi_1(k\frac{t}{n})}{20} \cdot \frac{t}{n}$$

The sum that appears is a Cauchy sum for the function $\frac{\phi_1}{20}$ on the interval $[0, n\Delta t]$. Otherwise said, it lies between a lower Riemann sum (associated with the partition of $[0, n\Delta t]$ into k equal intervals) and the corresponding upper sum. This suggests the formula ($t = n\Delta t$): Cauchy sum Riemann sums

[3]Instead of $|\hat{\phi}(t) - \phi(t)| \quad \forall t \in [0,T]$. The calculations are analogous.

[4]There exist much better methods, but this one has the property of being very simple.

$$\phi_1(t) = \phi_1(0) - \int_0^t \frac{\phi_1(s)}{20}\, ds \quad \forall t > 0 \tag{5}$$

It is interesting to notice that the two formulations (4) and (5) are equivalent, even if (5) requires (*apparently*!) less regularity for ϕ_1: for (5) it is evidently enough that ϕ_1 is a continuous function (if ϕ_1 is a continuous function, then $\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{\phi_1(k\frac{t}{n})}{20} \cdot \frac{t}{n} = \int_0^t \frac{\phi_1(s)}{20}\, ds$), while in (4) it is needed that ϕ_1 is differentiable. But this difference is not substantial: namely, if a continuous function verifies (5), then it is also differentiable (just use the fundamental theorem of integral calculus). Nonetheless, the formulation (5) happens to be useful for the proof of the theorem of existence and uniqueness, since this apparent weaker regularity assumption makes the proof simpler (more precisely, I am referring to the so-called method of successive approximations, which is *one* of the methods that can be used for the proof).

are equivalent

just a whimsical idea, then?

no, it works well for theorem proving...

The most relevant aspect of the approach via integral equation lies precisely in the request for *less regularity a priori* for the solution. If, in our small problem of the tank we can prove that the two formulations (Cauchy problem and integral equation) are equivalent, this fact is *not always true*. Clearly, we cannot describe the motion of a pool ball[5] by means a differential equation (more precisely, not in the elementary setting studied in the basic mathematical courses): the law of motion for sure is not described by differentiable functions! Having a reformulation of the problem as an integral equation, is an important picklock that allows us to unhinge (pardon, model) situations like this. Another ultra-classical example is given by the study of the trajectories of light rays when they cross bodies of different density: if we want to describe the trajectory of a light ray through the atmosphere, taking into account its density variations, we can do it by means of a trajectory that can be represented, parametrically, by means of derivable functions; if we want to describe the light ray that arrives to us from the part of a stick plunged into the lake, we cannot do it by means of derivable functions!

... but, above all, it allows for *less* regularity assumptions imposed *a priori* on the solution

final moral on this theme

I would like to make some further considerations on other aspects, which are relevant but often neglected. These aspects are three and are inextricably tied: notations, what is a differential equation and which are the data of a differential equation

Let's start with notations. A Cauchy problem as the one we have seen is usually described in many ways. Here I will show three among the most common, but there are more:

ahi! the notations!

$$\begin{cases} y' &= -\frac{y}{20} \\ y(0) &= 50 \end{cases} \qquad \begin{cases} \dot{x} &= -\frac{x}{20} \\ x(0) &= 50 \end{cases} \qquad \begin{cases} \frac{dx}{dt} &= -\frac{x}{20} \\ x(0) &= 50 \end{cases}$$

[5] In the case in which the ball hits at least one of the walls...

I will pay attention to the leftmost notation, to which I will refer from now on. A generic Cauchy problem will be then written as follows:

$$\begin{cases} y' & = & f(x,y) \\ y(x_0) & = & y_0 \end{cases}$$

Let's look at the differential equation, which is the most problematic part. *What is a differential equation?* It is a **PROBLEM**. Whose solution is a function. To be more specific, given the differential equation $y' = f(x,y)$, we shall say that a function $\phi : I \to \mathbb{R}$ (I is an open interval of $\mathbb{R}$) solves it if it is true that[6]:

a differential equation is a **PROBLEM**

$$\phi'(x) = f(x, \phi(x)) \quad \forall x \in I$$

It should be evident that the traditional notation is ambiguous, since it does not make clear enough the fact that "at the place of y" it has to be substituted the value of the "aspiring solution", evaluated at the point x. But this is the tradition[7].

the traditional notation is ambiguous

We have been speaking of notations and of "what is a differnetial equation". If it is true that a differential equation is a "shorthand" to point at the fact that we have a *problem* to solve, then we should expect that, as every problem, it has **DATA**. On the other hand, I have already mentioned them when I was talking of "continuous dependence from data". But, perhaps, we should be quite precise. In a Cauchy problem:

Which are the **DATA** of the problem?

$$\begin{cases} y' & = & f(x,y) \\ y(x_0) & = & y_0 \end{cases}$$

the data are:
- x_0, that is the initial point
- y_0, that is the initial value
- f, that is: the "law" that "governs" the dynamics of the system, and which is a real valued function, defined on a subset A of $\mathbb{R}^2$

initial point

initial value

the "law"

Notice that there is also an obvious compatibility condition on the data: the point (x_0, y_0) must belong to A.

compatibility on the data

For example, in our problem:

$$\begin{cases} y' & = & -\frac{y}{20} \\ y(0) & = & 50 \end{cases}$$

[6] If f is not defined on all of $\mathbb{R}^2$, it must be added that $(x, \phi(x))$ must belong, for all $x \in I$, to the domain of f

[7] As always, when tradition in involved, we cannot think that it became the tradition just by chance. But this kind of considerations would bring us too afar.

the data are:
- $x_0 = 0$
- $y_0 = 50$
- $f(x, y) = -\frac{y}{20}$

One further remark concerns the validity domain for the differential equation. This is clearly a subset of the domain of f. For example, the differential equation:

where the differential equation is meaningful?

$$y' = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

is clearly meaningless outside $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Notice that A is precisely the domain where "f" is defined.

Sometimes there could be *further restrictions*. Precisely in our example, it not too much meaningful to imagine a *negative* amount of liters of the liquid L_1 contained in the tank! So, we can think that the data of our problem was not the function $f(x, y) = -y/20$, but its *restriction* to the set $A = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Notice that, given this, we have also a compatibility condition, i.e. it must be $y_0 > 0$ (that is, in the tank there is some liquid L_1...). To tell the truth, there could be further conditions. Our model could not be suited if the quantity of liquid L_1 were too small (statistical fluctuations, possibly due to thermal motion, could play a role, for very small values of L_1), and so our model is not valid for values of y smaller than some positive threshold.

Further restrictions. Traditionally forgotten...

please, be always on the alert!

I stop here, but clearly we could proceed further. I emphasize that in our problem we have assumed that time can be considered as a continuous variable: it seems quite reasonable, but in other cases it could be not appropriate (or useful). Further, one could ask whether one is not seriously compelled to make the model "more complicate[8]". For example, consider stochastic aspects (in our case it does not seem too plausible, since even after one hour of "dilution" it is not plausible that the quantity of liquid L_1 is so small that one has to consider the aspects that were mentioned above) or consider spatial aspects in the interaction (in our case it could be relevant to consider the diffusion of liquid L_2 in the solution[9]).

[8] I say "seriously", because anyone who has seen in his life some "scientific" works pseudo-applied (there is plenty of them, around!), knows that it is one of the easiest (and more useless) things to do.

[9] If I drop water into wine, the water does not diffuse instantaneously, nor uniformly. Even worse would be if we had, for example, a mixture of water and olive oil.

We came to the end.

But let us think for one moment more. Do you think that the way in which the problem was formulated was exhaustive? For me, **NO**. I hope that the flow of 5 liters per minute is constant (everything was pointing to that, but if it is not true we must re-start). Moreover: how is taken away the solution? At a constant rate? And from where? Maybe one has to take into account (or not) diffusion aspects, depending on the place where the "hole" is (it will be really a hole? At the top? Or at the bottom?). I could go on for a while.

Summing up, the given problem was a "bookish" problem[10]. It was not a *true* problem. If you really want to know how to write a differential equation, take a *real* problem and work!

Fioravante PATRONE
Dipartimento di Ingegneria della Produzione,
Termoenergetica e Modelli Matematici
P.le Kennedy - Pad D
16129 Genova - ITALY
patrone@diptem.unige.it

http://www.diptem.unige.it/patrone homepage
http://tdg.dima.unige.it web teaching
http://www.citg.unige.it/citg.htm web server "CITG"
http://www.scallywag.it web page del gruppo Scaλλywag

[10]It was bookish also for a characteristic quite anti-educational, that is found in 99% and more of the "problems" proposed by the huge majority of the books. It is paid a lot of attention to choose problems that are solvable with the methods that one just learnt from the "theory". In the real world life is not so easy (and boring).