Introduction to Game Theory and Applications

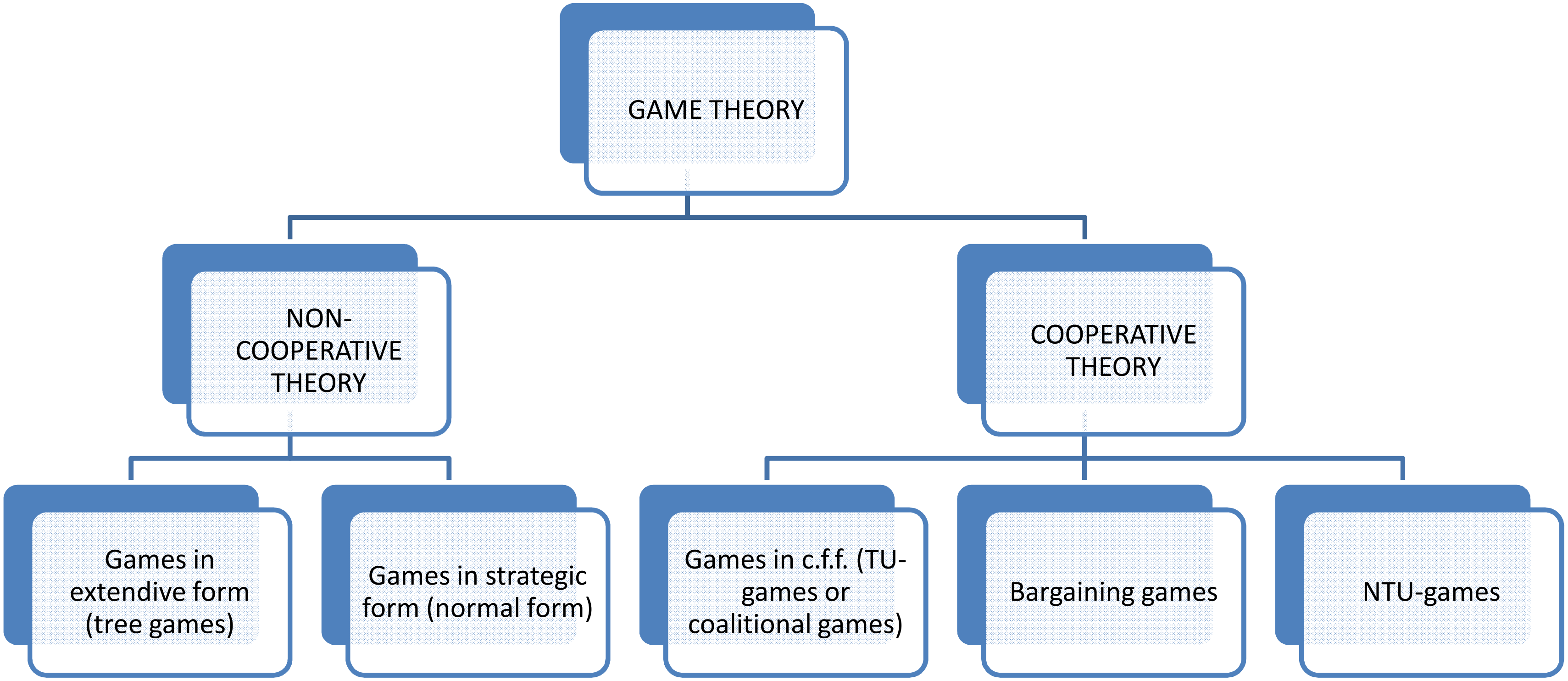
Stefano MORETTI and Fioravante PATRONE

LAMSADE (CNRS), Paris Dauphine

and

DIPTEM, University of Genoa

Paris, Telecom ParisTech, 2010



- Dominant strategies
- Nash eq. (NE)
- Subgame perfect NE
- NE & refinements

...

- Core
- Shapley value
- Nucleolus
- τ-value
- PMAS

....

- Nash sol.
- Kalai-Smorodinsky

....

- CORE
- NTU-value
- Compromise value

...

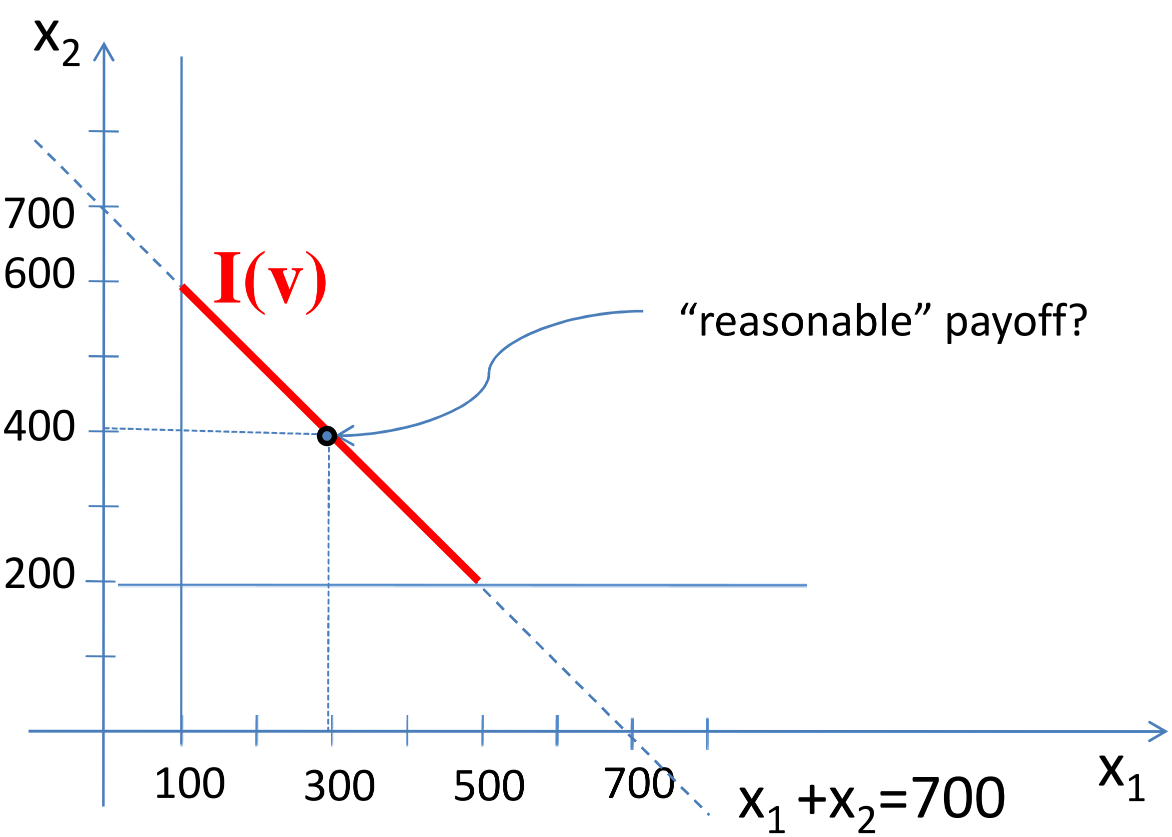
No binding agreements
No side payments
Q: Optimal behaviour in conflict situations

binding agreements
side payments are possible (sometimes)
Q: Reasonable (cost, reward)-sharing

Simple example

Alone, player 1 (singer) and 2 (pianist) can earn 100€ 200€ respect.
Together (duo) 700€

How to divide the (extra) earnings?



Imputation set: $I(v) = \{x \in \mathbb{R}^2 | x_1 \geq 100,\ x_2 \geq 200,\ x_1 + x_2 = 700\}$

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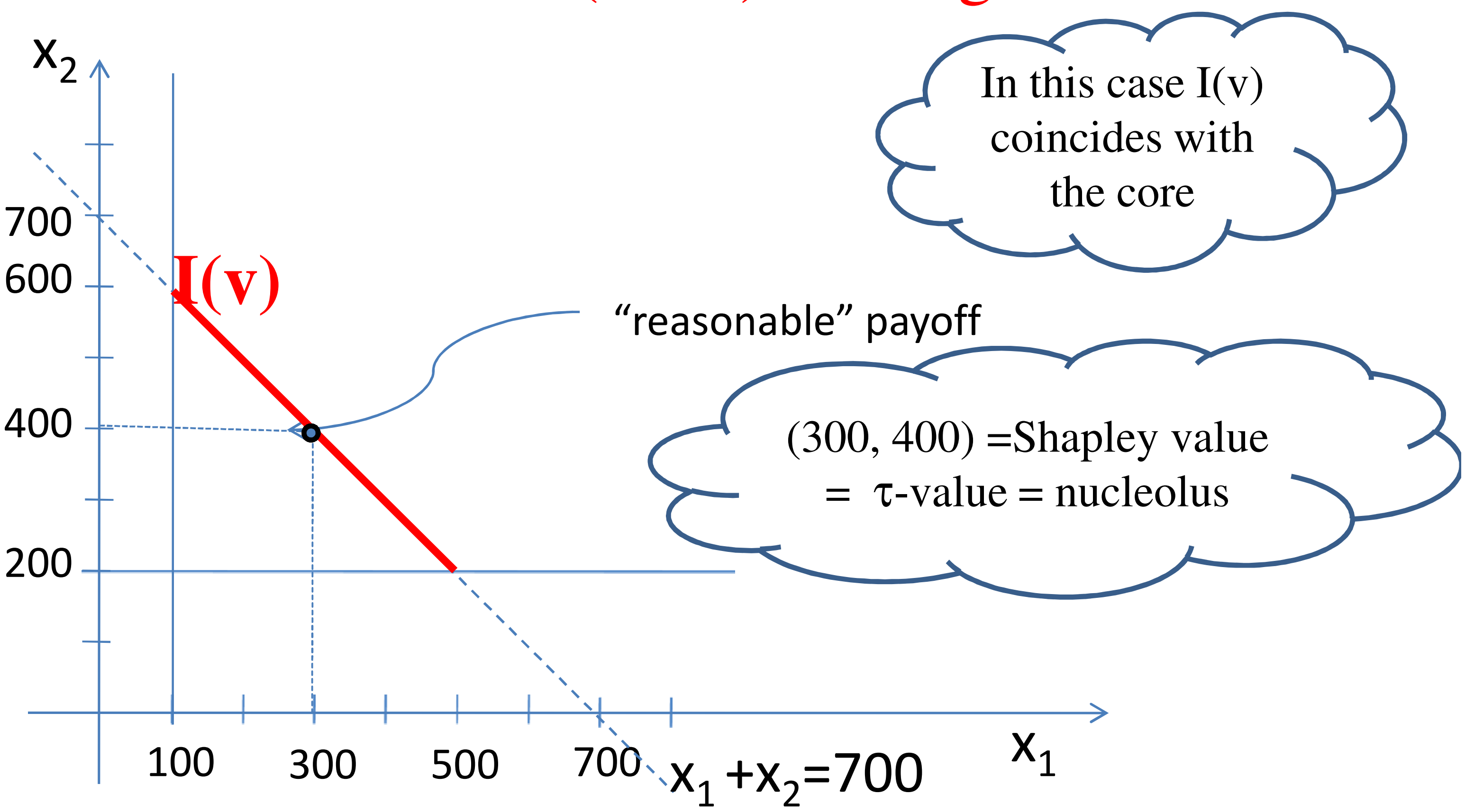
Together (duo) 700€

How to divide the (extra) earnings?



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COOPERATIVE GAME THEORY

Games in coalitional form

TU-game: (N,v) or v

$N=\{1, 2, \ldots, n\}$	set of players
$S \subset N$	coalition
2^N	set of coalitions

DEF. v: $2^N \rightarrow \mathbb{R}$ with $v(\emptyset)=0$ is a Transferable Utility (TU)-game with player set N.

NB: $(N,v) \leftrightarrow v$

NB2: if $n=|N|$, it is also called n-person TU-game, game in colaitional form, coalitional game, cooperative game with side payments...

v(S) is the value (worth) of coalition S

Example

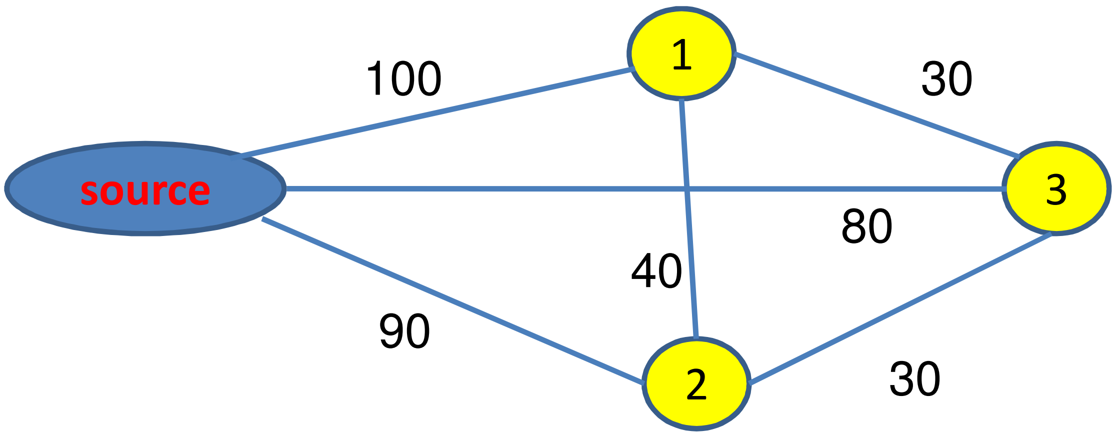
(Glove game) $N=L \cup R$, $L \cap R = \emptyset$

$i \in L$ ($i \in R$) possesses 1 left (right) hand glove

Value of a pair: 1€

$v(S)=\min\{|L \cap S|, |R \cap S|\}$ for each coalition $S \in 2^N \setminus \{\emptyset\}$.

Example
(Three cooperating communities)

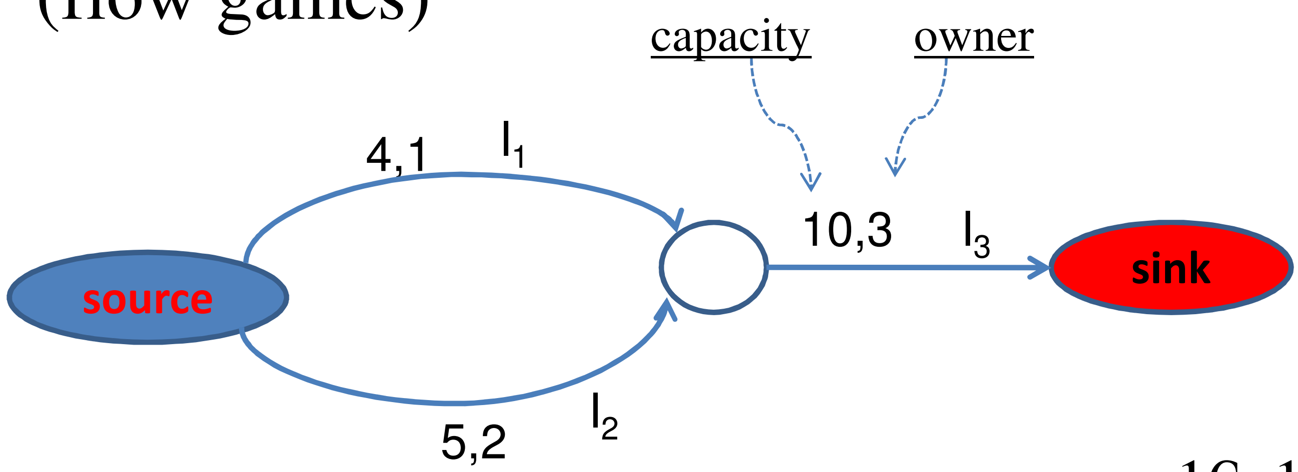


$N=\{1,2,3\}$

S=	∅	{1}	{2}	{3}	{1,2}	{1.3}	{2,3}	{1,2,3}
c(S)	0	100	90	80	130	110	110	140
v(S)	0	0	0	0	60	70	60	130

$$v(S)=\Sigma_{i\in S}c(i) - c(S)$$

Example
(flow games)



N={1,2,3}

1€: 1 unit source → sink

S=	∅	{1}	{2}	{3}	{1,2}	{1.3}	{2,3}	{1,2,3}
v(S)	0	0	0	0	0	4	5	9

DEF. (N,v) is a superadditive game iff

$v(S \cup T) \geq v(S) + v(T)$ for all S,T with $S \cap T = \emptyset$

Q.1: which coalitions form?

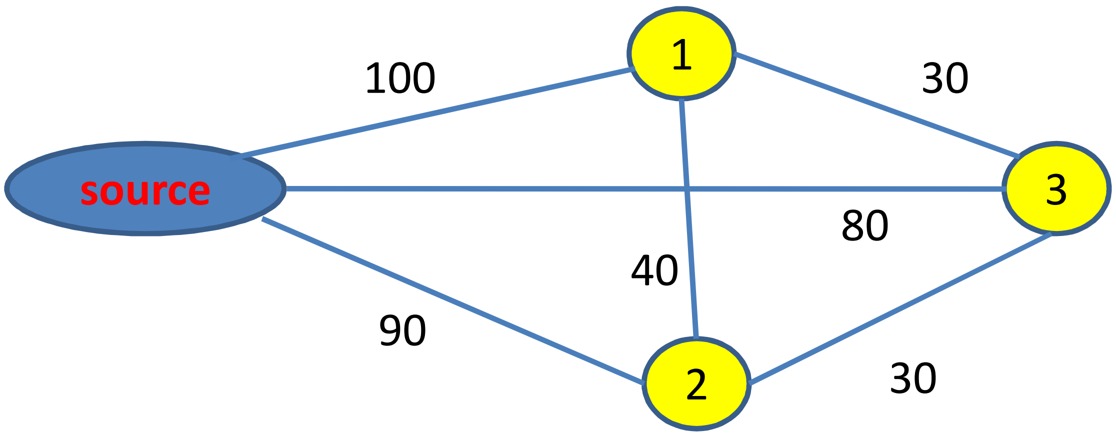
Q.2: If the grand coalition N forms, how to divide v(N)?
(how to allocate costs?)

Many answers! (solution concepts)

One-point concepts:
- Shapley value (Shapley 1953)
- nucleolus (Schmeidler 1969)
- τ-value (Tijs, 1981)
- …

Subset concepts:
- Core (Gillies, 1954)
- stable sets (von Neumann, Morgenstern, ’44)
- kernel (Davis, Maschler)
- bargaining set (Aumann, Maschler)
- …..

Example
(Three cooperating communities)



N={1,2,3}

S=	∅	{1}	{2}	{3}	{1,2}	{1.3}	{2,3}	{1,2,3}
c(S)	0	100	90	80	130	110	110	140
v(S)	0	0	0	0	60	70	60	130

$$v(S)=\Sigma_{i\in S}c(i) - c(S)$$

Show that v is superadditive and c is subadditive.

Claim 1: (N,v) is superadditive

We show that $v(S \cup T) \geq v(S)+v(T)$ for all $S,T \in 2^N \setminus \{\varnothing\}$ with $S \cap T = \varnothing$

60=v(1,2)≥v(1)+v(2)=0+0

70=v(1,3)≥v(1)+v(3)=0+0

60=v(2,3)≥v(2)+v(3)=0+0

60=v(1,2)≥v(1)+v(2)=0+0

130=v(1,2,3) ≥v(1)+v(2,3)=0+60

130=v(1,2,3) ≥v(2)+v(1,3)=0+70

130=v(1,2,3) ≥v(3)+v(1,2)=0+60

Claim 2: (N,c) is *subadditive*

We show that $c(S \cup T) \leq c(S)+c(T)$ for all $S,T \in 2^N \setminus \{\varnothing\}$ with $S \cap T = \varnothing$

130=c(1,2) ≤ c(1)+c(2)=100+90

110=c(2,3) ≤ c(2)+v(3)=100+80

110=c(1,2) ≤ c(1)+v(2)=90+80

140=c(1,2,3) ≤ c(1)+c(2,3)=100+110

140=c(1,2,3) ≤ c(2)+c(1,3)=90+110

140=c(1,2,3) ≤ c(3)+c(1,2)=80+130

Example

(Glove game) (N,v) such that $N = L \cup R$, $L \cap R = \varnothing$

$v(S) = 10 \min\{|L \cap S|, |R \cap S|\}$ for all $S \in 2N \setminus \{\varnothing\}$

Claim: the glove game is superadditive.

Suppose $S, T \in 2^N \setminus \{\varnothing\}$ with $S \cap T = \varnothing$. Then

$v(S)+v(T) = \min\{|L \cap S|, |R \cap S|\} + \min\{|L \cap T|, |R \cap T|\}$

$= \min\{|L \cap S|+|L \cap T|, |L \cap S|+|R \cap T|, |R \cap S|+|L \cap T|, |R \cap S|+|R \cap T|\}$

$\leq \min\{|L \cap S|+|L \cap T|, |R \cap S|+|R \cap T|\}$

since $S \cap T = \varnothing$

$= \min\{|L \cap (S \cup T)|, |R \cap (S \cup T)|\}$

$= v(S \cup T)$.

The imputation set

DEF. Let (N,v) be a n-persons TU-game.
A vector $x=(x_1, x_2, \ldots, x_n) \in IR^N$ is called an imputation iff

(1) x is individual rational i.e.

$$x_i \geq v(i) \text{ for all } i \in N$$

(2) x is efficient

$$\Sigma_{i \in N}\, x_i = v(N)$$

[interpretation x_i: payoff to player i]

$$I(v)=\{x \in IR^N \mid \Sigma_{i \in N}\, x_i = v(N),\ x_i \geq v(i) \text{ for all } i \in N\}$$

Set of imputations

Example

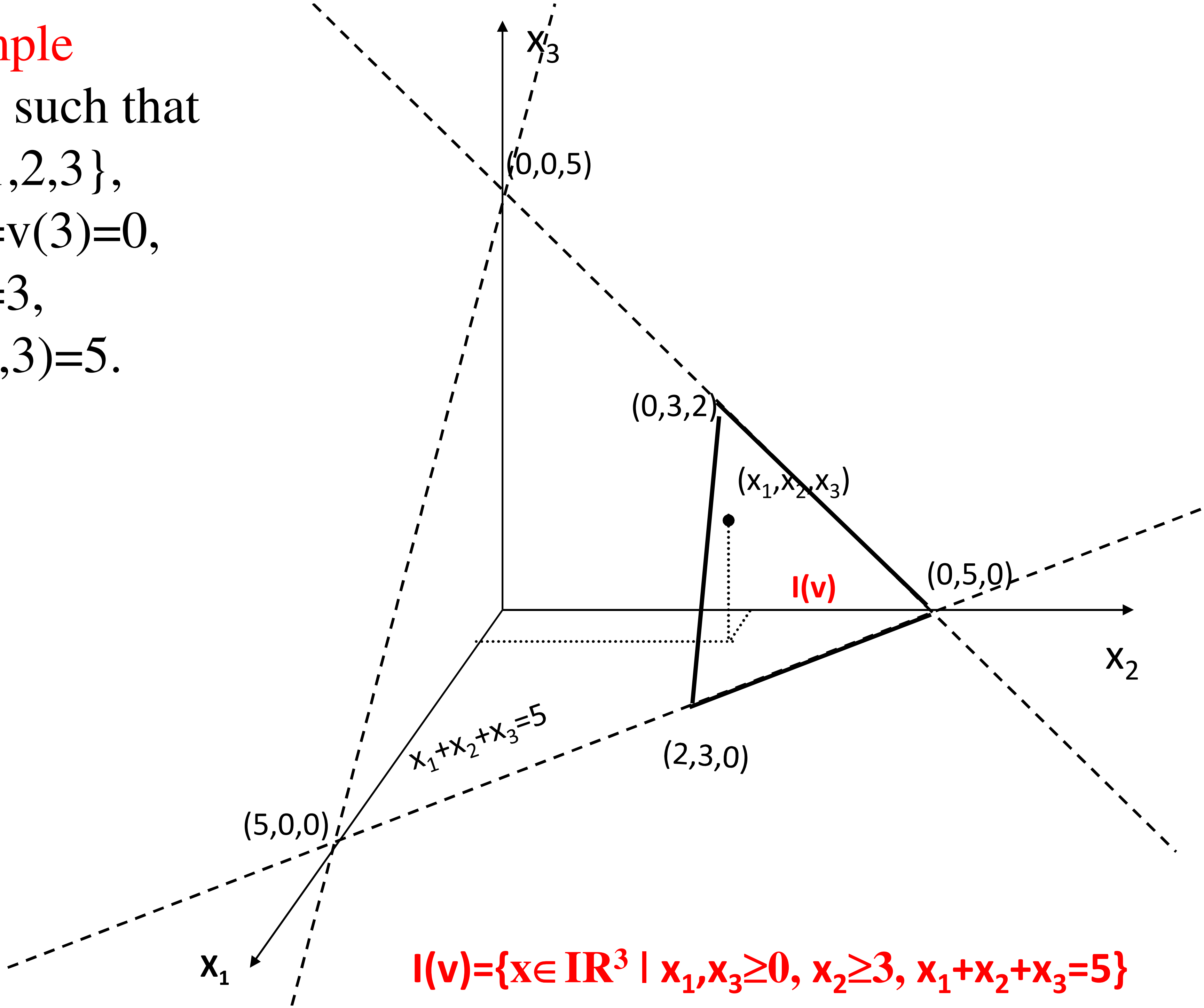
(N,v) such that

N={1,2,3},

v(1)=v(3)=0,

v(2)=3,

v(1,2,3)=5.



$I(v)=\{x\in IR^3 \mid x_1,x_3\geq 0,\ x_2\geq 3,\ x_1+x_2+x_3=5\}$

Claim: (N,v) a n-person (n=|N|) TU-game. Then

$$I(v) \neq \emptyset \quad \Leftrightarrow v(N) \geq \sum_{i \in N} v(i)$$

Proof

($\Rightarrow$)

Suppose $x \in I(v)$. Then

$$v(N) \underset{\text{EFF}}{=} \sum_{i \in N} x_i \underset{\text{IR}}{\geq} \sum_{i \in N} v(i)$$

($\Leftarrow$)

Suppose $v(N) \geq \sum_{i \in N} v(i)$. Then the vector

$$(v(1), v(2), \ldots, v(n-1), \underbrace{v(N) - \sum_{i \in \{1,2,\ldots,n-1\}} v(i)}_{\geq v(n)})$$

is an imputation.

The core of a game

DEF. Let (N,v) be a TU-game. The core C(v) of (N,v) is the set

$C(v)=\{x\in I(v) \mid \Sigma_{i\in S}\, x_i \geq v(S) \text{ for all } S\in 2^N\setminus\{\emptyset\}\}$

stability conditions

no coalition S has the incentive to split off if x is proposed

Note: $x \in C(v)$ iff

(1) $\Sigma_{i\in N}\, x_i = v(N)$ *efficiency*

(2) $\Sigma_{i\in S}\, x_i \geq v(S)$ for all $S\in 2N\setminus\{\emptyset\}$ *stability*

Bad news: C(v) can be empty

Good news: many interesting classes of games have a non-empty core.

Example

(N,v) such that

$N=\{1,2,3\}$,

$v(1)=v(3)=0$,

$v(2)=3$,

$v(1,2)=3$,

$v(1,3)=1$

$v(2,3)=4$

$v(1,2,3)=5$.

Core elements satisfy the following conditions:

$x_1, x_3 \geq 0,\ x_2 \geq 3,\ x_1+x_2+x_3=5$

$x_1+x_2 \geq 3,\ x_1+x_3 \geq 1,\ x_2+x_3 \geq 4$

We have that

$5-x_3 \geq 3 \Leftrightarrow x_3 \leq 2$

$5-x_2 \geq 1 \Leftrightarrow x_3 \leq 4$

$5-x_1 \geq 4 \Leftrightarrow x_1 \leq 1$

$$C(v)=\{x \in IR^3 \mid 1 \geq x_1 \geq 0, 2 \geq x_3 \geq 0,\ 4 \geq x_2 \geq 3,\ x_1+x_2+x_3=5\}$$

Example

(N,v) such that

N={1,2,3},

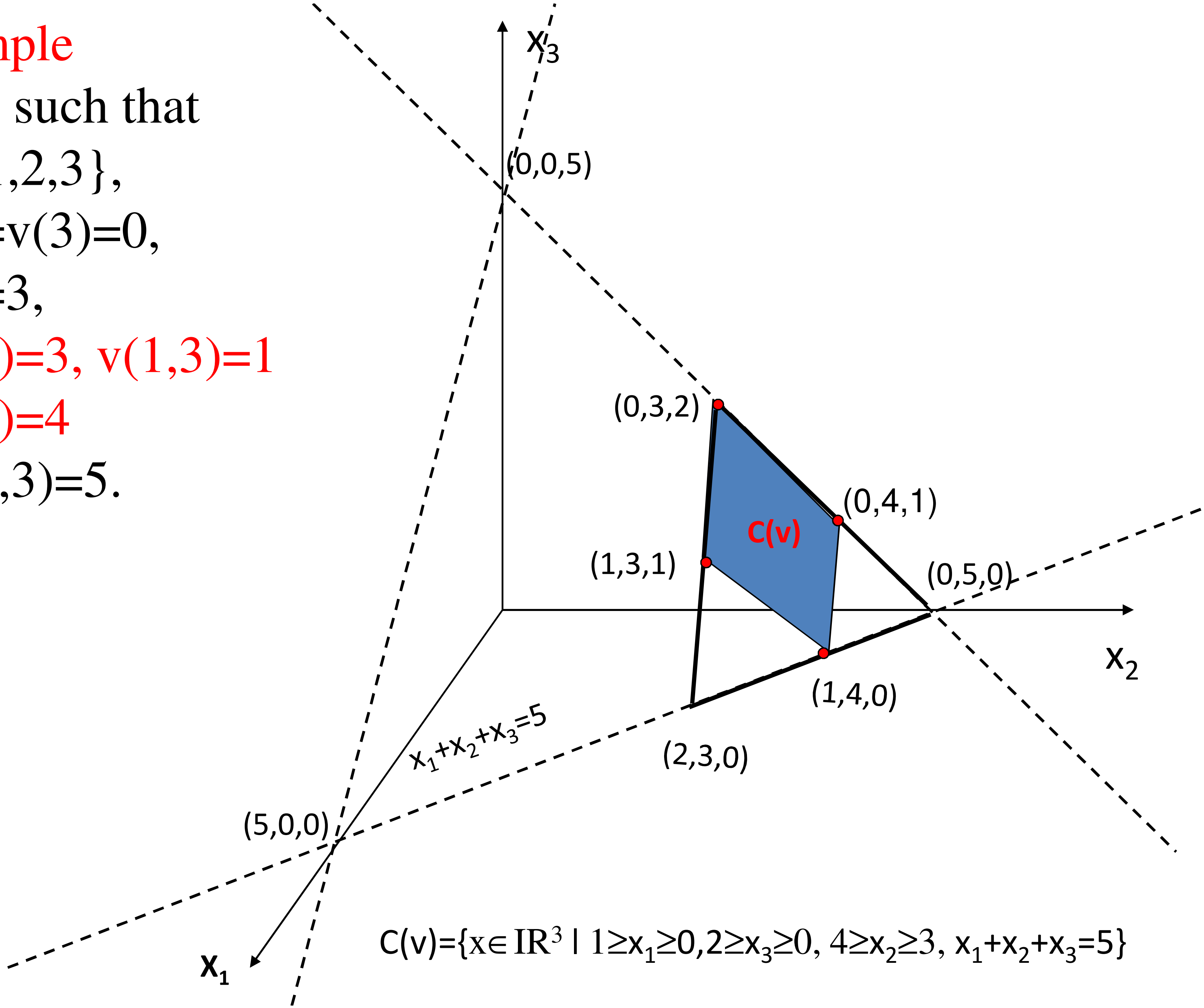
v(1)=v(3)=0,

v(2)=3,

v(1,2)=3, v(1,3)=1

v(2,3)=4

v(1,2,3)=5.



$C(v)=\{x\in IR^3 \mid 1\geq x_1\geq 0, 2\geq x_3\geq 0, 4\geq x_2\geq 3, x_1+x_2+x_3=5\}$

Example (Game of pirates) Three pirates 1,2, and 3. On the other side of the river there is a treasure (10€). At least two pirates are needed to wade the river…

(N,v), $N=\{1,2,3\}$, $v(1)=v(2)=v(3)=0$,

$v(1,2)=v(1,3)=v(2,3)=v(1,2,3)=10$

Suppose $(x_1, x_2, x_3) \in C(v)$. Then

$$
\begin{array}{ll}
\text{efficiency} & x_1 + x_2 + x_3 = 10 \\
\text{stability} & \left\{ \begin{array}{l} x_1 + x_2 \geq 10 \\ x_1 + x_3 \geq 10 \\ x_2 + x_3 \geq 10 \end{array} \right.
\end{array}
$$

$$20 = 2(x_1 + x_2 + x_3) \geq 30$$

Impossible. So $C(v)=\varnothing$.

Note that (N,v) is superadditive.

Example

(Glove game with L={1,2}, R={3})

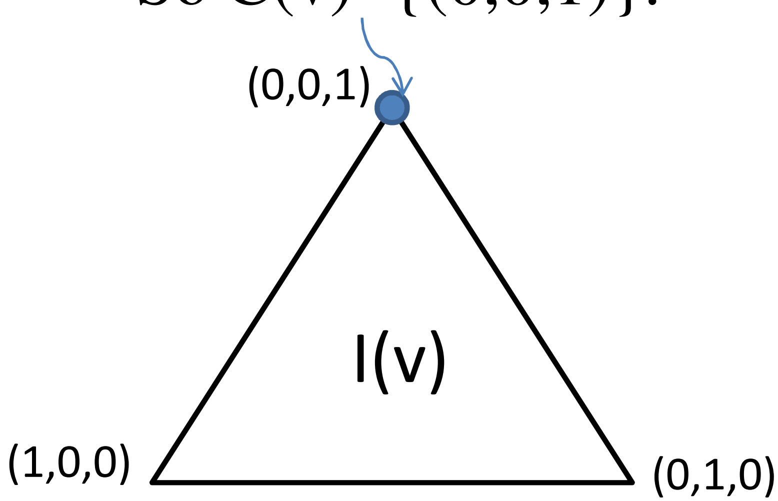
v(1,3)=v(2,3)=v(1,2,3)=1, v(S)=0 otherwise

Suppose $(x_1, x_2, x_3) \in C(v)$. Then

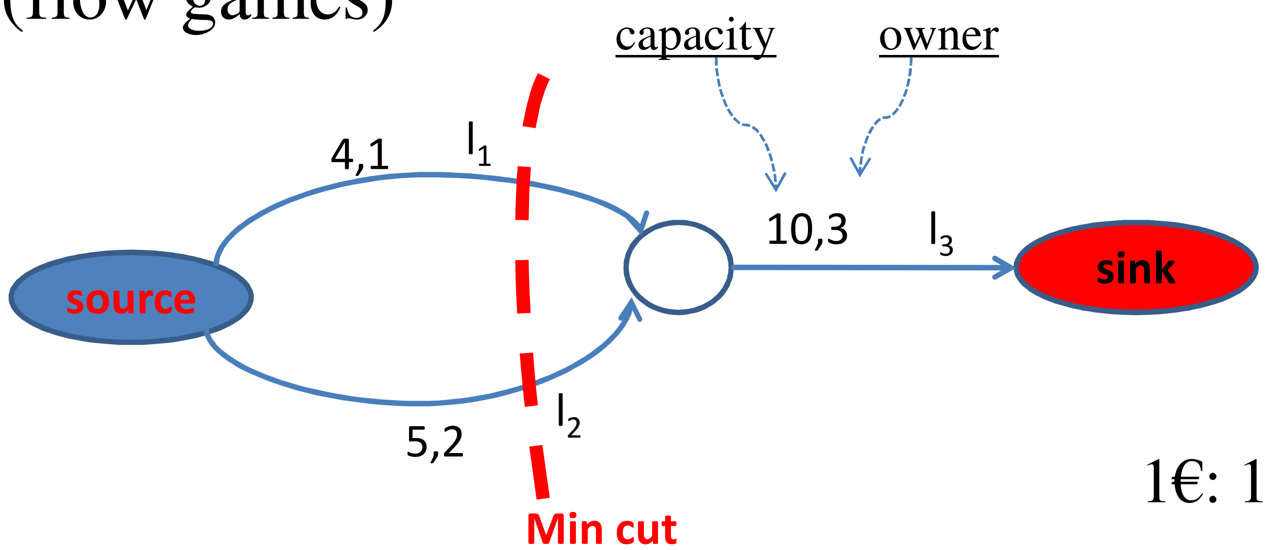
$$\left.\begin{aligned} & x_1 + x_2 + x_3 = 1 \\ & x_1 + x_3 \geq 1 \\ & x_2 \geq 0 \end{aligned}\right\} \quad \begin{aligned} & x_2 = 0 \\ & x_1 + x_3 = 1 \end{aligned}$$

$x_2 + x_3 \geq 1$ $\quad x_1 = 0$ and $x_3 = 1$

So C(v)={(0,0,1)}.



Example
(flow games)



N={1,2,3}

1€: 1 unit source → sink

S=	∅	{1}	{2}	{3}	{1,2}	{1.3}	{2,3}	{1,2,3}
v(S)	0	0	0	0	0	4	5	9

Min cut $\{l_1, l_2\}$. Corresponding core element (4,5,0)

Non-emptiness of the core

Notation: Let $S \in 2^N \setminus \{\emptyset\}$.

e^S is a vector with $(e^S)_i = \begin{cases} 1 \text{ if } i \in S \\ 0 \text{ if } i \notin S \end{cases}$

DEF. A collection $\mathbf{B} \subset 2^N \setminus \{\emptyset\}$ is a balanced collection if there exist $\lambda(S) > 0$ for $S \in \mathbf{B}$ such that:

$$e^N = \sum_{S \in \mathbf{B}} \lambda(S)\, e^S$$

Example: $N = \{1,2,3\}$, $\mathbf{B} = \{\{1,2\},\{1,3\},\{2,3\}\}$, $\lambda(S) =$ for $S \in \mathbf{B}$

$e^N = (1,1,1) = 1/2\ (1,1,0) + 1/2\ (1,0,1) + 1/2\ (0,1,1)$

Balanced games

DEF. (N,v) is a balanced game if for all balanced collections $\mathbf{B} \subset 2^N \setminus \{\emptyset\}$

$$\sum_{S \in \mathbf{B}} \lambda(S)\, v(S) \leq v(N)$$

Example: $N=\{1,2,3\}$, $v(1,2,3)=10$,

$v(1,2)=v(1,3)=v(2,3)=8$

(N,v) is not balanced

$1/2 v(1,2) + 1/2\, v(1,3) + 1/2\, v(2,3) > 10 = v(N)$

Variants of duality theorem

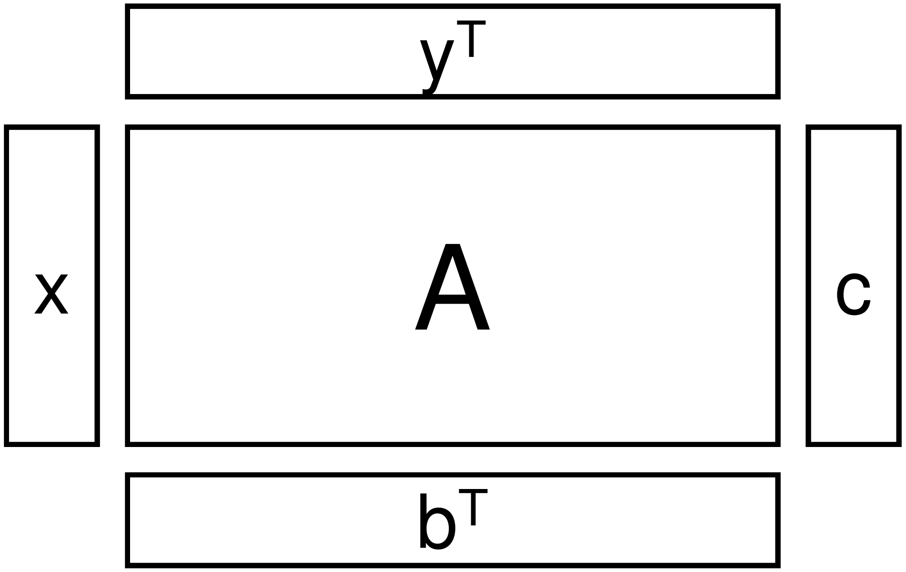
<u>Duality theorem.</u>

$\min\{x^T c \mid x^T A \geq b\}$

||

$\max\{b^T y \mid Ay = c,\ y \geq 0\}$

(if both programs feasible)

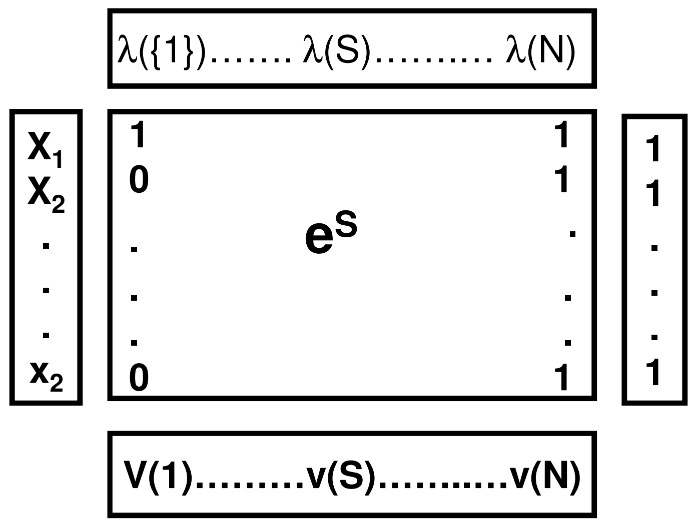


Bondareva (1963) | Shapley (1967)
Characterization of games with non-empty core

Theorem

$$(N,v) \text{ is a balanced game} \Leftrightarrow C(v)\neq\emptyset$$

Proof First note that $x^T e^S = \sum_{i\in S} x_i$



$C(v)\neq\emptyset$

$\Updownarrow$

$v(N)=\min\{x^T e^N \mid x^T e^S \geq v(S) \quad \forall S\in 2^N\backslash\{\emptyset\}\}$

$\Updownarrow$ duality

$v(N)=\max\{v^T\lambda \mid \sum_{S\in 2^N\backslash\{\emptyset\}}\lambda(S)e^S=e^N,\ \lambda\geq 0\}$

$\Updownarrow$

$\forall\lambda\geq 0, \quad \sum_{S\in 2^N\backslash\{\emptyset\}}\lambda(S)e^S=e^N \Rightarrow v^T\lambda=\sum_{S\in 2^N\backslash\{\emptyset\}}\lambda(S)v(S)\leq v(N)$

$\Updownarrow$

(N,v) is a balanced game

Convex games (1)

DEF. An n-persons TU-game (N,v) is convex iff

$$v(S)+v(T)\leq v(S\cup T)+v(S\cap T) \quad \text{for each } S,T\in 2^N.$$

This condition is also known as *submodularity*. It can be rewritten as

$$v(T)-v(S\cap T)\leq v(S\cup T)-v(S) \quad \text{for each } S,T\in 2^N$$

For each $S,T\in 2^N$, let $C=(S\cup T)\setminus S$. Then we have:

$$v\big(C\cup(S\cap T)\big)-v(S\cap T)\leq v(C\cup S)-v(S)$$

Interpretation: the marginal contribution of a coalition C to a disjoint coalition S does not decrease if S becomes larger

Convex games (2)

- It is easy to show that submodularity is equivalent to

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

for all $i \in N$ and all $S,T \in 2^N$ such that $S \subseteq T \subseteq N \setminus \{i\}$

- interpretation: player's marginal contribution to a large coalition is not smaller than her/his marginal contribution to a smaller coalition (which is stronger than superadditivity)
  - Clearly all convex games are superadditive ($S \cap T = \emptyset \ldots$)
  - A superadditive game can be not convex (try to find one)
- An important property of convex games is that they are (*totally*) balanced, and it is "easy" to determine the core (coincides with the Weber set, i.e. the convex hull of all marginal vectors…)

Example

(N,v) such that

N={1,2,3},

v(1)=v(3)=0,

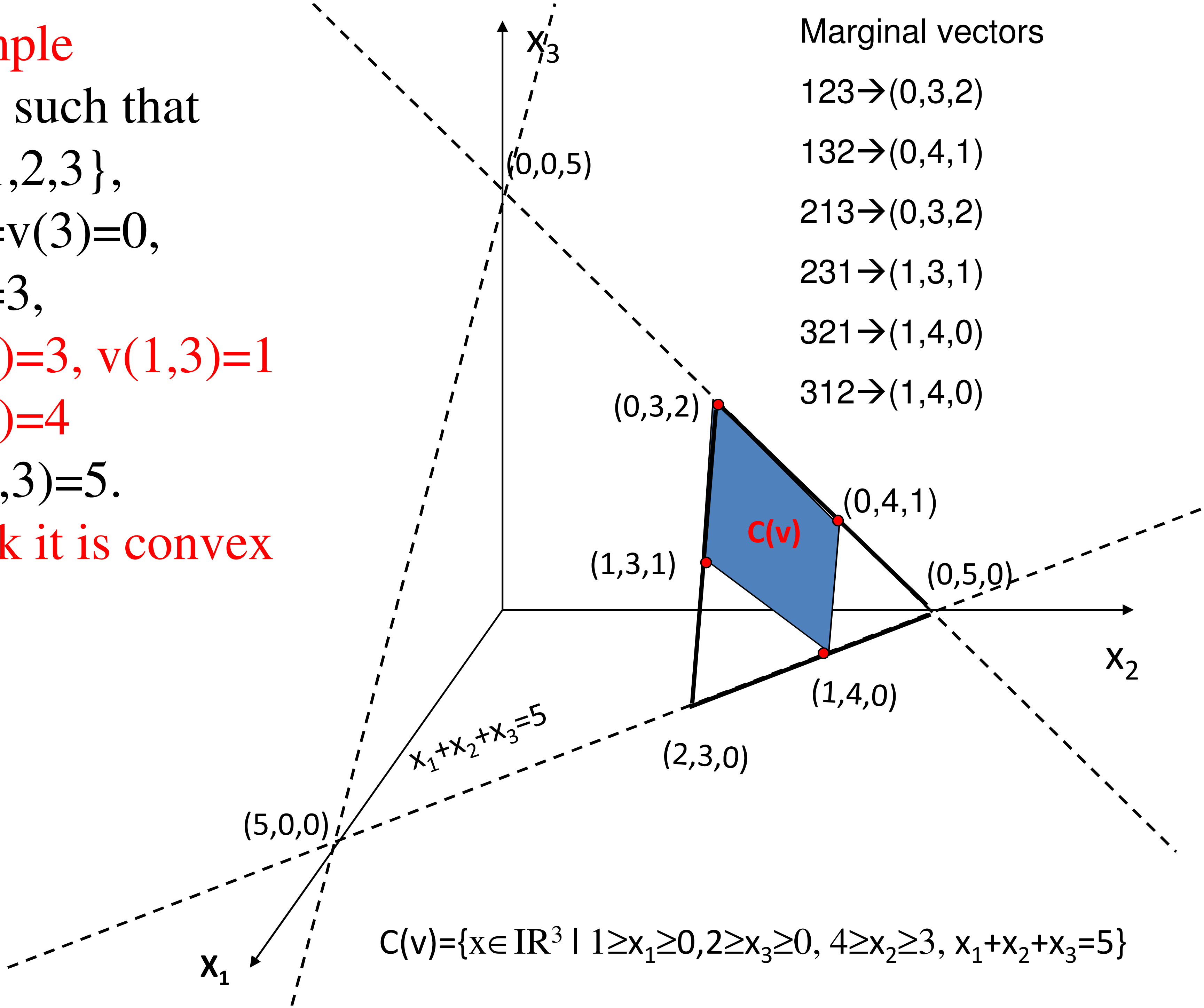
v(2)=3,

v(1,2)=3, v(1,3)=1

v(2,3)=4

v(1,2,3)=5.

Check it is convex



Marginal vectors

123→(0,3,2)

132→(0,4,1)

213→(0,3,2)

231→(1,3,1)

321→(1,4,0)

312→(1,4,0)

$C(v)=\{x\in IR^3 \mid 1\geq x_1\geq 0, 2\geq x_3\geq 0, 4\geq x_2\geq 3, x_1+x_2+x_3=5\}$

Operations Research (OR)

- Analysis of situations in which **one decision maker**, guided by an objective function, faces an **optimization problem**.
- OR focuses on the question of **how to act** in an optimal way and, in particular, on the issues of *computational complexity* and the design of *efficient algorithms*.

OR and GT→ORG

OPERATIONS RESEARCH GAMES

- Basic (discrete) structure of a graph, network or system that underlies various types of **combinatorial optimization** problems.
- Assumes that at least two players are located at or control parts (e.g., vertices, edges, resource bundles, jobs) of the underlying system.
- A cooperative game can be associated with this type of optimization problem.

Scheduling problems

- In this category: *sequencing game*, *permutation game*, *assignment game.*
- Games whose characteristic function depends from the position of players in a queue.
- Players can be seen as sellers of their initial position and buyers of their final position.

Permutation situation <N,A>

- N={1,...,n} set of agents and A processing cost matrix NxN;
- Each agent has one job and one machine that can process a job
- Each machine is <u>allowed</u> to process at most one job
- Each machine is <u>able</u> to process every possible job
- If player *i* processes its own job on the machine of player *j*, then the cost of the process is a_{ij} (element of A row *i* and column *j*).

Permutation problem

Optimization problem:

- Which job must be assigned to which machine in order to minimize the cost of the process?
- In other words, how to maximize the savings with respect the situation in which each agent processes its job on its own machine?

Permutation game

- Given a permutation situation <N,A>
- The permutation game (N,v) is defined as the TU-game with
  - N as the set of players
  - And the characteristic function is such that

$$v(S)= \sum_{i\in S} a_{ii} - \min_{p\in \Pi_S} \sum_{i\in S} a_{ip(i)}$$

for each $S\in 2^N \setminus \{\emptyset\}$ (obviously by definition $v(\emptyset)=0$) and Π_S is the set of all permutations of the lements of S.

- The worth v(S) represents the maximum saving that S can obtain thanks to an optimal program with respect the program where each agents works with its own machine.

Example: Consider a permutation situation where N={1,2,3} and A is such that

$$A=\begin{pmatrix} 8 & 4 & 2 \\ 2 & 4 & 10 \\ 5 & 6 & 10 \end{pmatrix}$$

The corresponding permutation game (N,v) is represented in the following table

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
v(S)	0	0	0	6	11	0	12
Optimal permutation→	p*=(1)	p*=(2)	p*=(3)	p*=(2,1)	p*=(3,1)	p*=(2,3)	p*=(3,1,2)
	8-8=0	4-4=0	10-10=0	12-6=6	18-7=11	14-14=0	22-10=12

Note that v({1,2,3})-v({1,3})=1<6=v({1,2})-v({1}) which implies that permutation games are not convex.

Notes on Permutation games

- permutation games are totally balanced.
- A particular class of permutation games are the *assignment game* introduced by Shapley e Shubik in 1971.
  - Such games are inspired to two-sided markets in which non-divisible goods are exchanged with money (model used for private market of used cars, auctions etc.)

Production problems (Owen (1975))

- In this category: *linear production games, flow games*.
- Players may produce a product.
- Each coalition can use a set of technologies (linear) which allow the coalition to transform a resource bundle in a vector of products.
- The market can absorbs whatever amount of products at a given price (which is independent of the quantities produced).

Linear Production Situation <N,P,G,A,B,c> where

- $N=\{1,\ldots,n\}$ player set
- $G=(G_1, G_2,\ldots, G_q)$ vector of resources that can be used to produce consumption goods (products)
- $P=(P_1, P_2,\ldots, P_m)$ vector of products
- $A\geq 0$ production matrix with *m* rows and *q* columns: for the production of $\alpha\geq 0$ units of product P_j it is required αa_{j1} units of resource G_1, αa_{j2} units of resource G_2 etc.
- $B=(b^1, b^2,\ldots, b^n)$ where $b^i\in IR^q$ for each $i\in N$ is the *resource bundle* of player i (quantity of each resource in G own by player i).
- $c^T=(c_1, c_2,\ldots, c_m)$ vector of fixed market price of products.

Linear Production Problem

- Given a resource bundle $b \in IR^q$, a *feasible production plan* may be described as a vector $x \in IR^m$ such that $x^T A \leq b$
  - interpretation: produce for each $j \in \{1,2,\ldots,m\}$ x_j units of product P_j.
- The *profit* of a production plan is then given by the product $x^T c$;
- Problem: find the feasible production plan that maximize the profit, given the resource bundle b

$$\text{profit}(b) = \max\{x^T c \mid x \geq 0,\ x^T A \leq b\}$$

Linear Production (LP) game

DEF. Let <N,P,G,A,B,c> be a a linear production situation. The associated LP game is the n-person TU-game (N,v) such that the worth v(S) of coalition S is given by the solution of the LP problem where the resource bundle is the sum of the resource bundles of players in S, in formula

$$v(S)= \text{profit}(\sum_{i\in S} b_i)=\max\{x^T c \mid x\geq 0,\ x^T A\leq \sum_{i\in S} b_i\}$$

for each $S\subseteq 2^N\backslash\{\emptyset\}$ (by convention $v(\emptyset)=0$).

Example: Consider an LP situation with three players N={1,2,3}, two resources, two products and A,B and c as in the following:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},\ b_1 = \begin{pmatrix} 5 \\ 8 \end{pmatrix},\ b_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix},\ b_3 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \text{ and } c^T = \begin{pmatrix} 5 & 7 \end{pmatrix}$$

The corresponding LP game is the one shown in the following table

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
v(S)	23	14	0	40	25	19	42
	profit	profit	profit	profit	profit	profit	profit
Resource bundle of Coalition S	$b = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$	$b = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$	$b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$	$b = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$	$b = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$	$b = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$	$b = \begin{pmatrix} 10 \\ 12 \end{pmatrix}$

Results on LP games

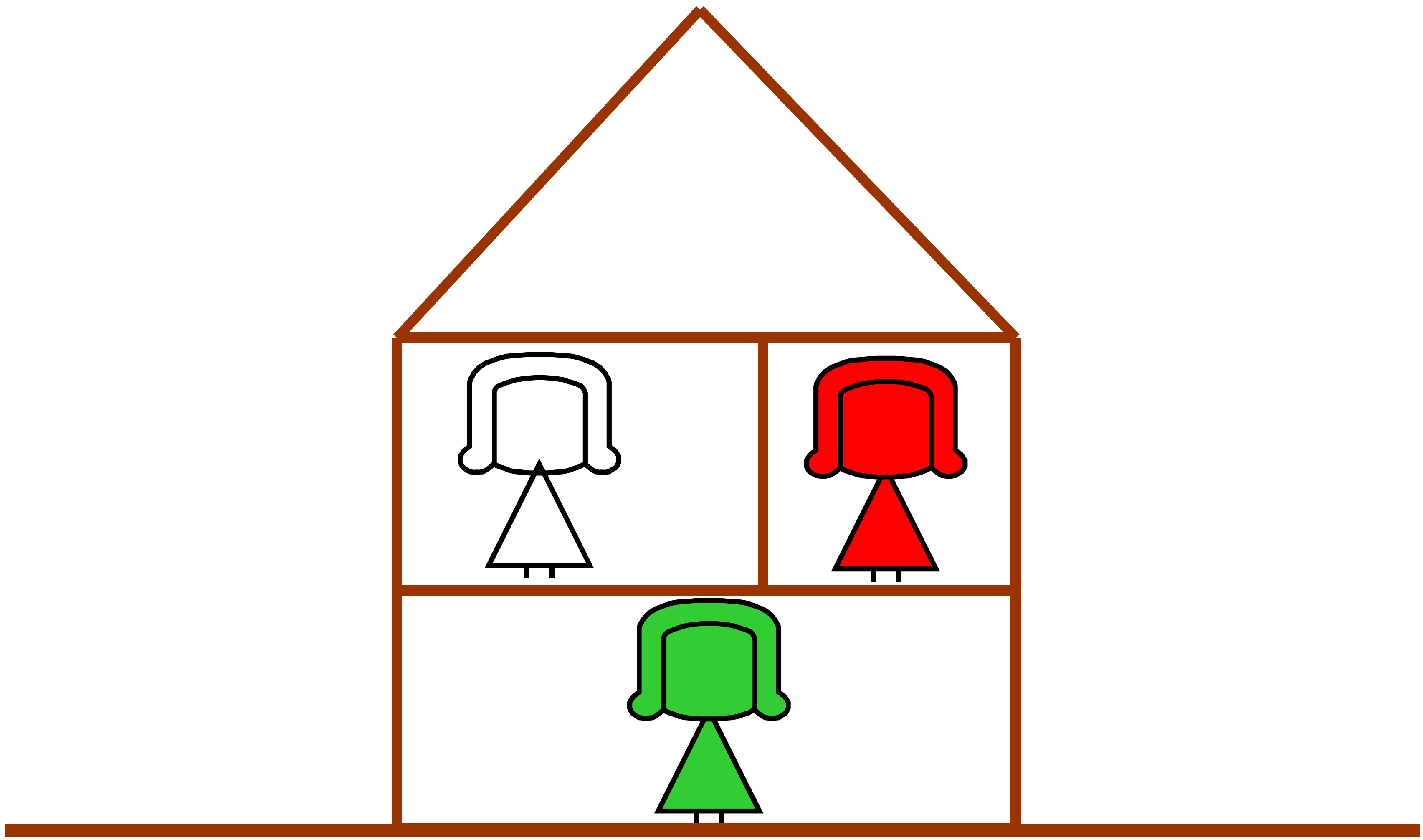
- It is possible to prove that LP games are totally balanced.
- To find a core allocation, first solve the dual problem of the LP problem, that is find the vector y* (shadow) which solves the dual problem

$$\min\{(\textstyle\sum_{i\in S} b_i)y \mid y\geq 0,\ Ay\geq c\}$$

- The allocation obtained as $z_i = b_i y^*$ for eavh $i\in N$ is in the core of LP game (N,v).

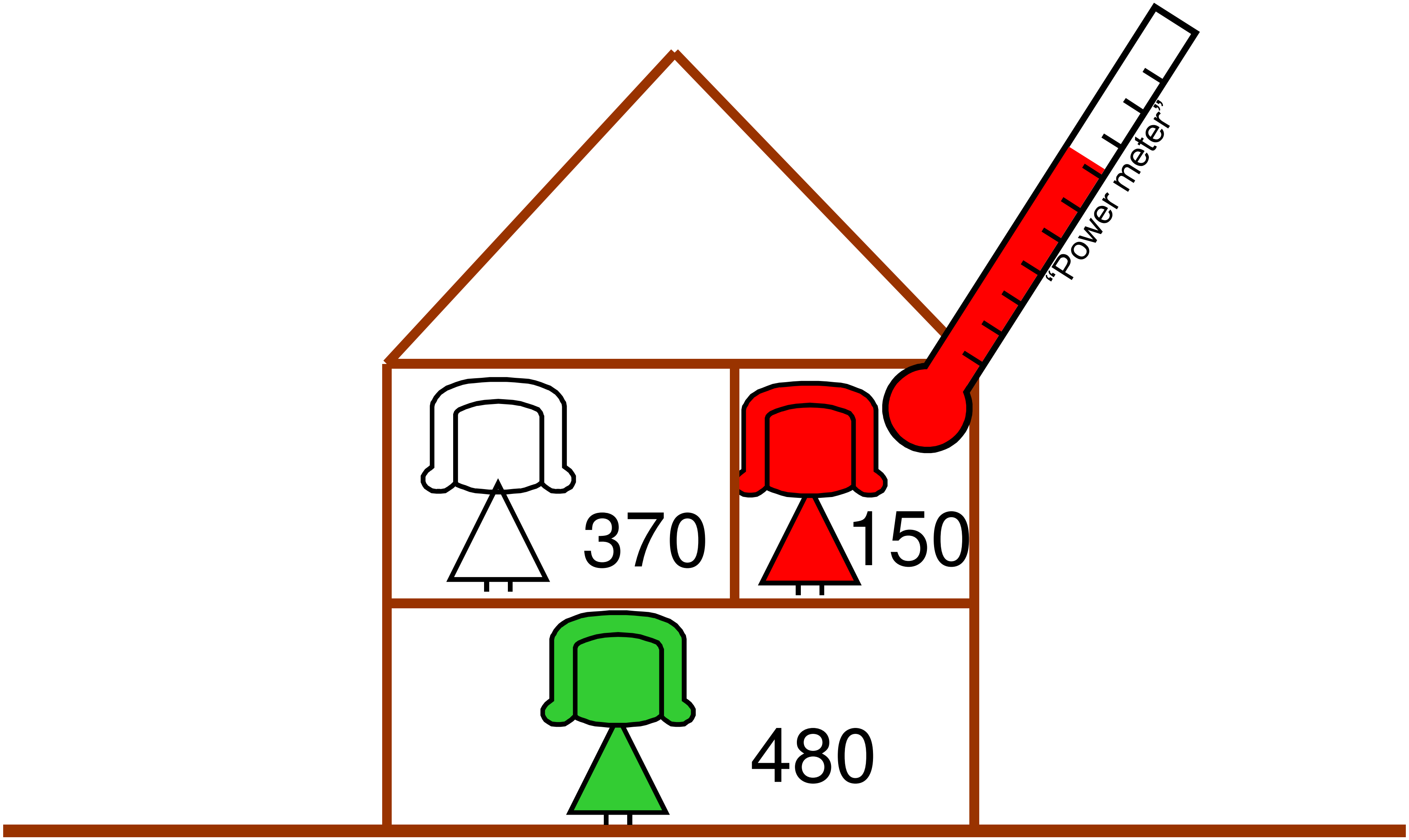
Example: in the previous example $y^*=(3,1)$, and therefore $z_1=5\times 3+8\times 1=23$, $z_2=17$ and $z_3=2$.

Sometimes there is nothing to divide…

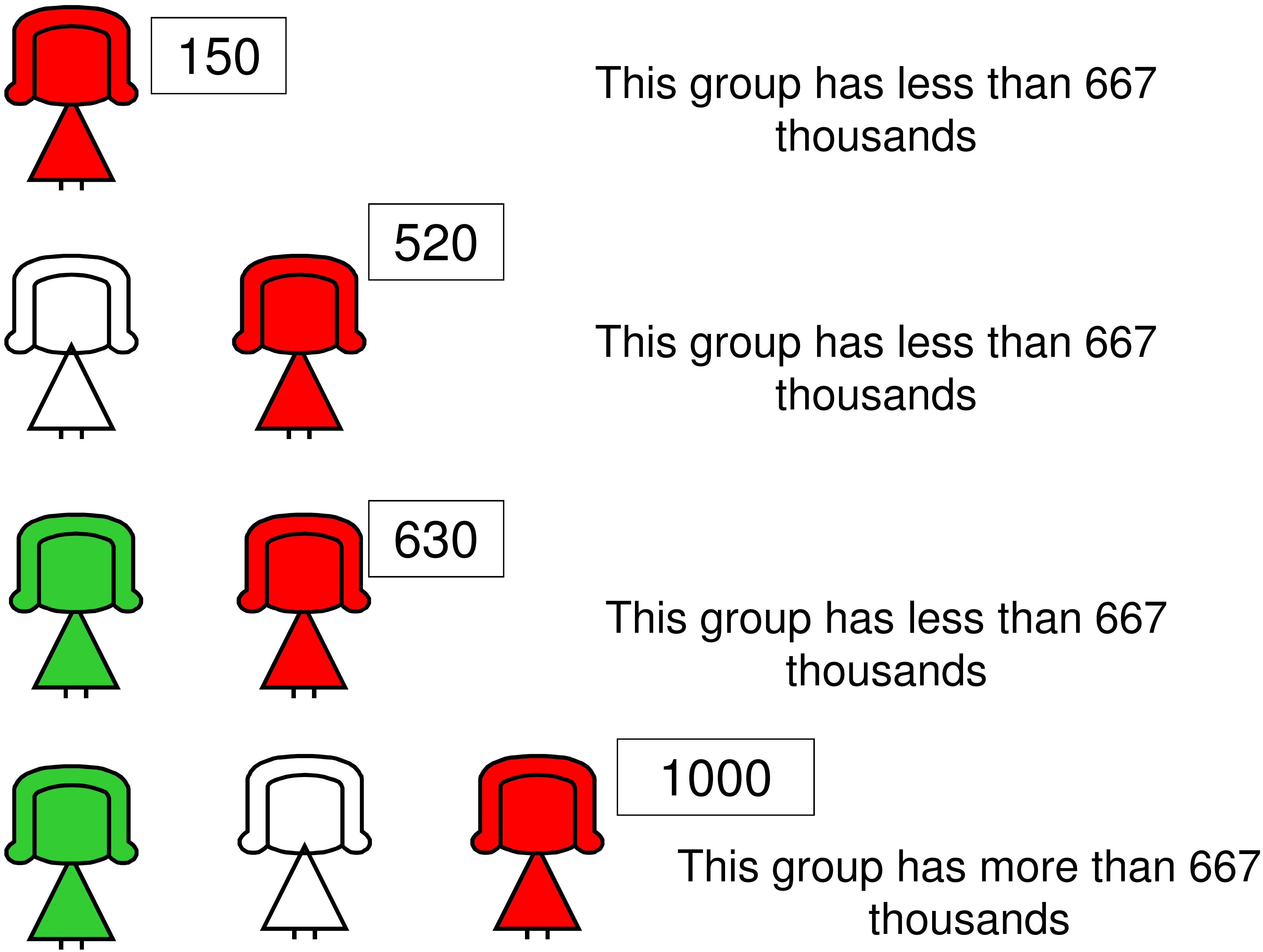


- Each owner in the jointly-owned building has a weight (in thousandths)
- Decision rule: to take a decision concerning the common facilities (e.g. to build an elevator) a group with at least 667 thousandths is winning
- How to measure the power of each owner?

Power index



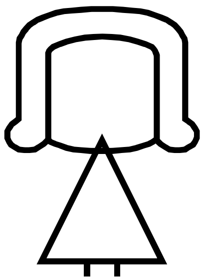
Which properties should a power index satisfy?


150
This group has less than 667 thousands
520
This group has less than 667 thousands
630
This group has less than 667 thousands
1000
This group has more than 667 thousands

0

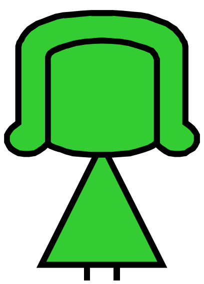
This group has less than 667 thousands

370

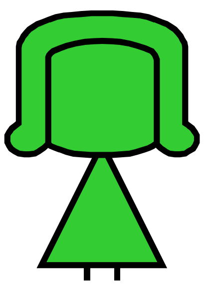


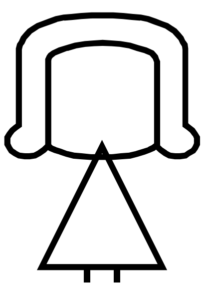
This group has less than 667 thousands

480



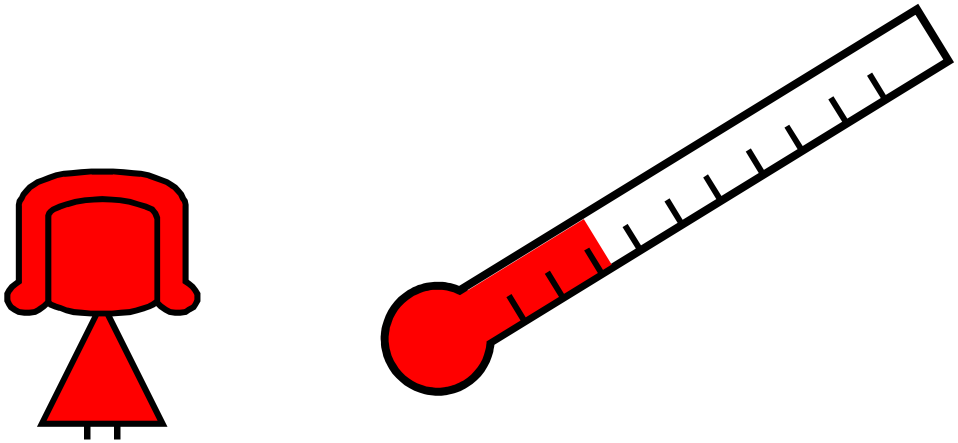
This group has less than 667 thousands





850

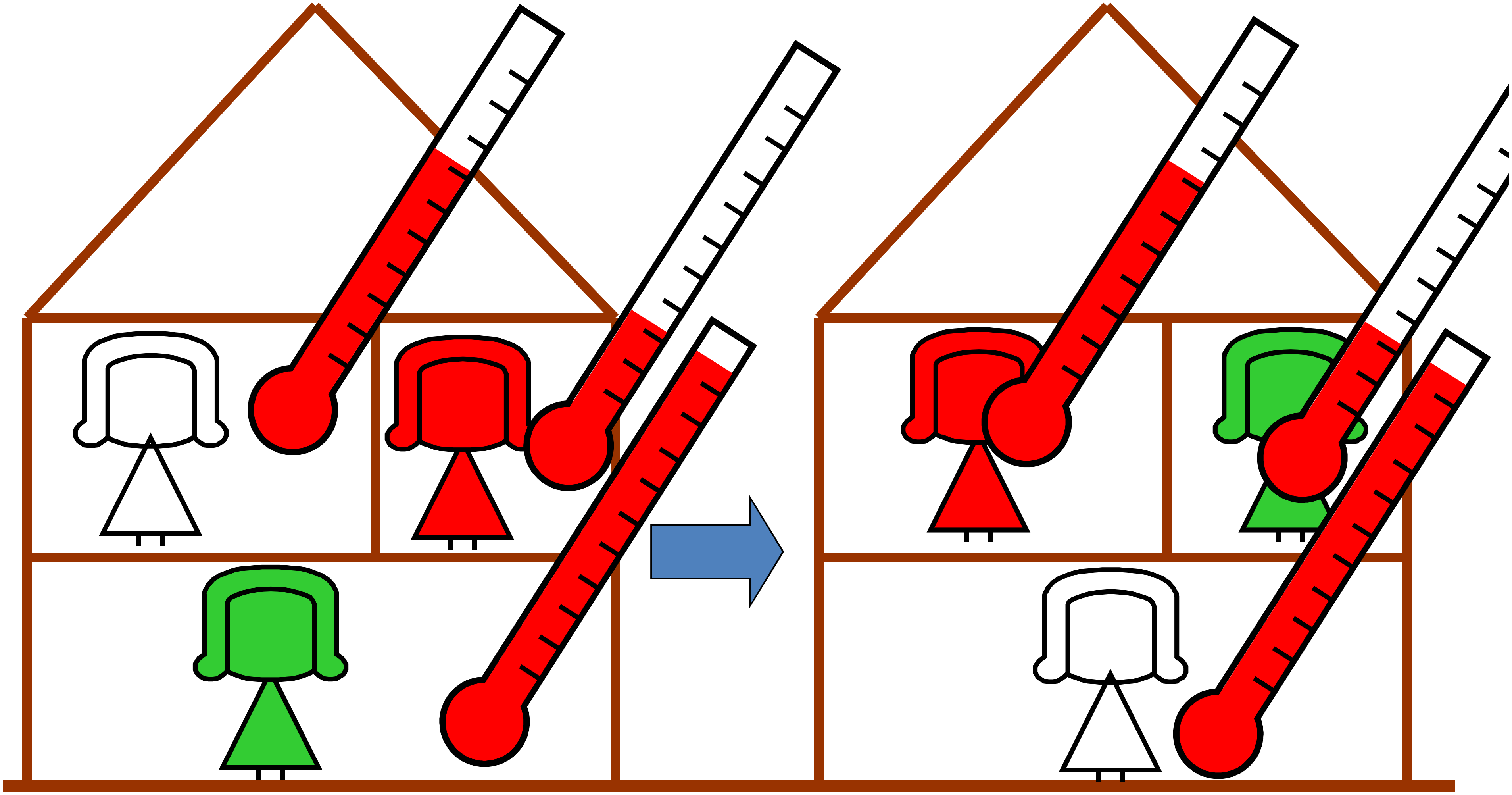
This group has more than 667 thousands



= 0

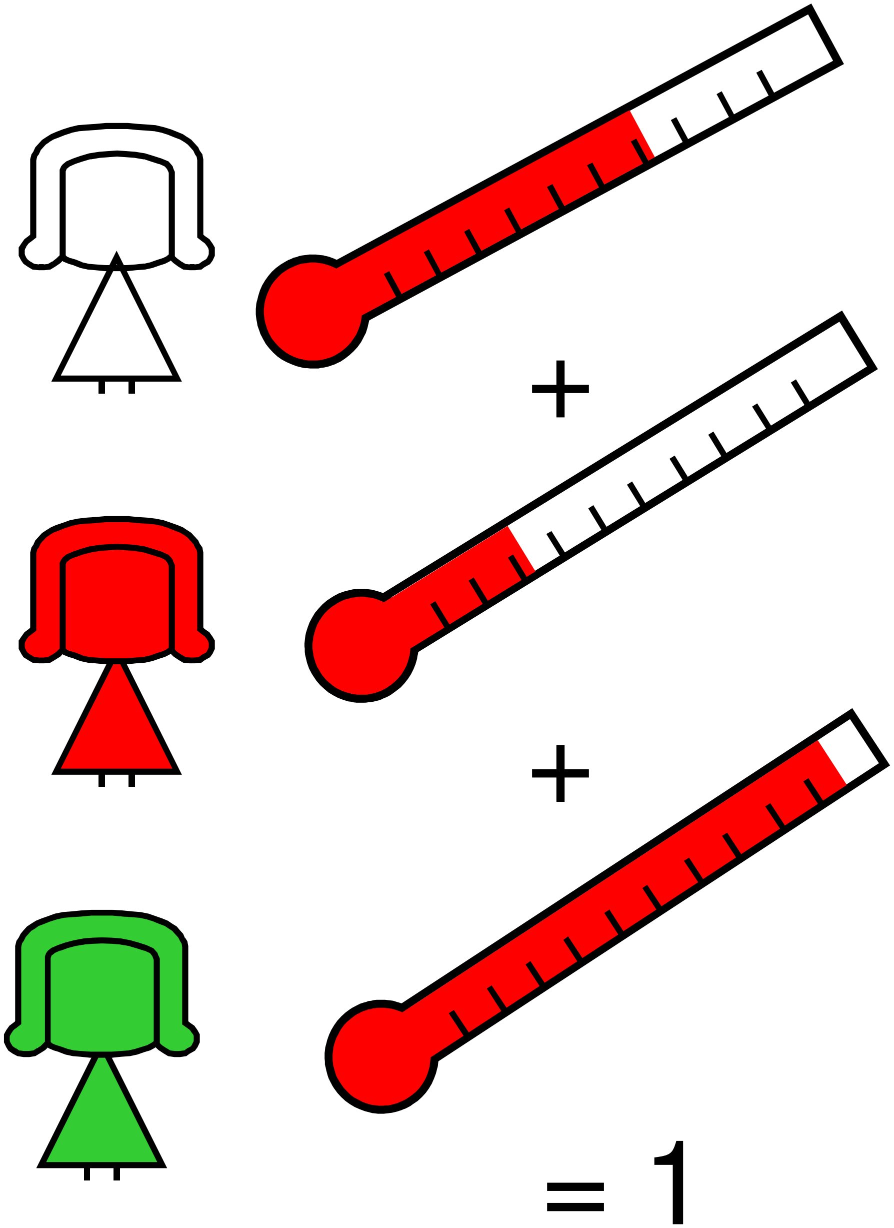
Null player property:

The power of the owners who never contribute to make a winning group must be zero.



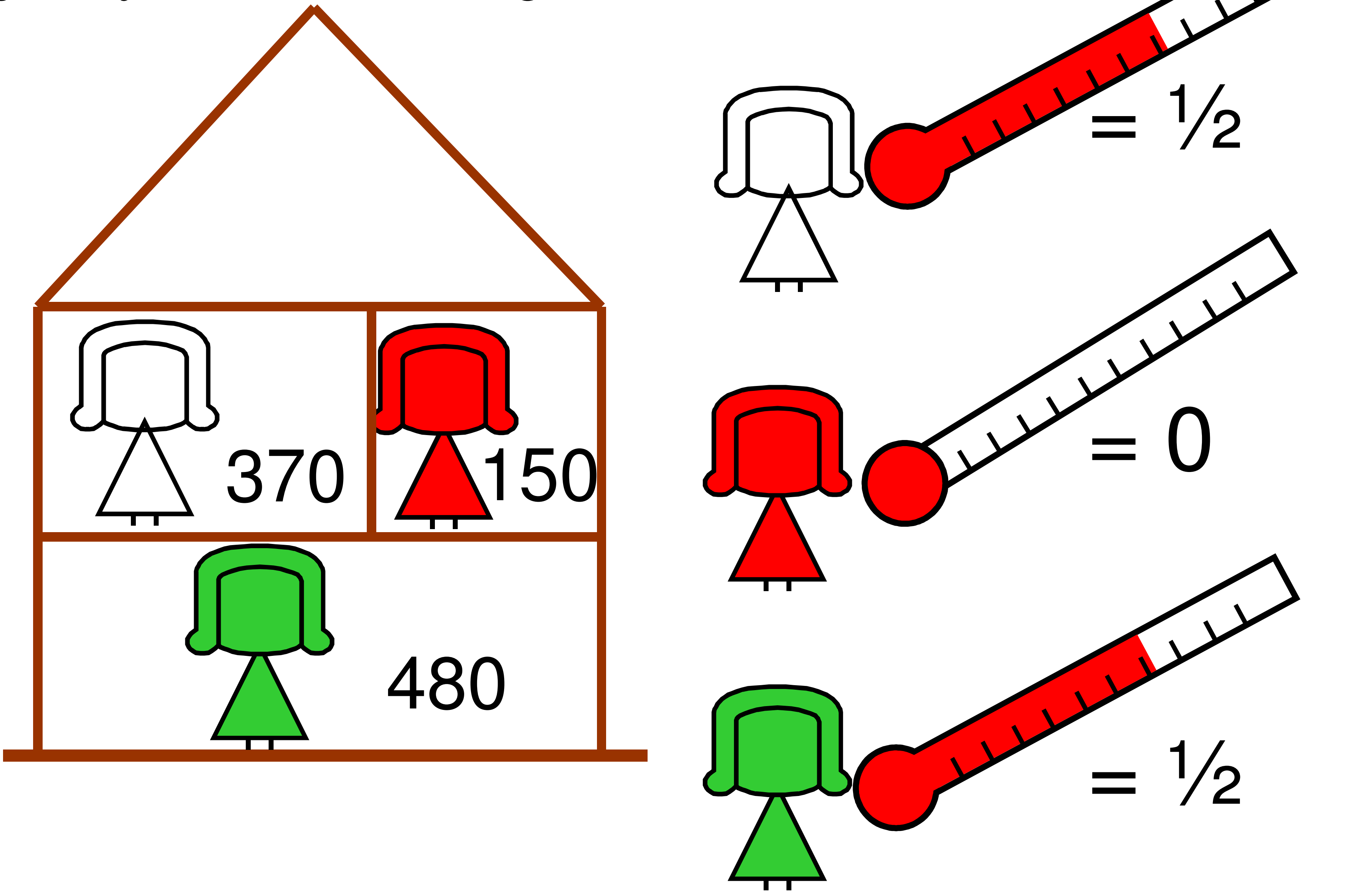
Anonimity property:

The power index should not depend on the names of the owners



<u>Efficiency property</u>: the sum of the powers must be 1

… a power index which satisfies such properties in the jointly-owned building



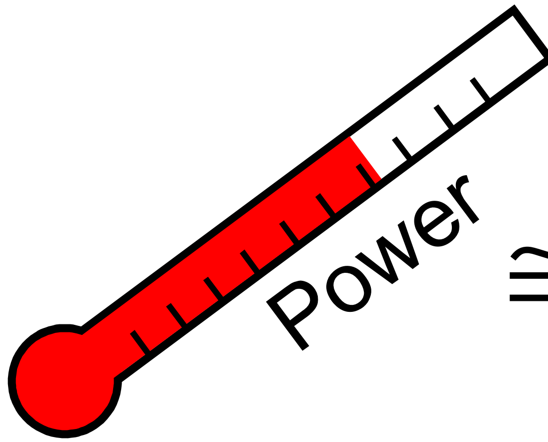
UN Security Council decisions

- Decision Rule: **substantive resolutions** need the positive vote of at least nine Nations but…

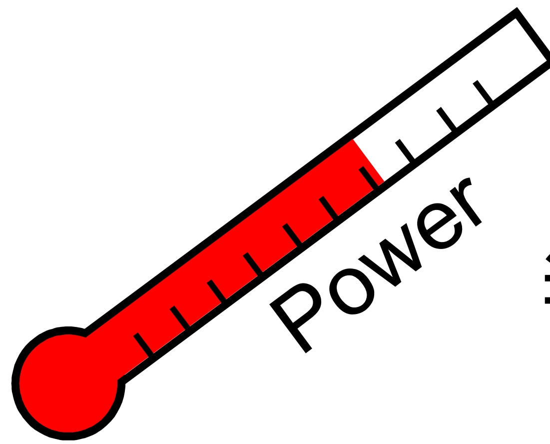
  …it is sufficient the negative vote of one among the permanent members to reject the decision.
- How much decision power each Nation inside the ONU council to force a substantive decision?
- Game Theory gives an answer using the Shapley-Shubik power index:

UN Security Council

- 15 member states:
  - 5 Permanent members: China, France, Russian Federation, United Kingdom, USA
  - 10 temporary seats (held for two-year terms) (http://www.un.org/)



≅ 19.6%



≅ 0.2%

temporary seats since January 1st 2007
until January 1st 2009

Simple games

DEF. A TU-game (N,v) is a *simple* game iff

$v(S) \in \{0,1\}$ for each coalition $S \in 2^N$ and $v(N)=1$

Example (weighted majority game)

The administration board of acompany is formed by three stockholders 1,2, and 3 with 55%, 40% and 5% of shares, respectively.

To take a decision the majority is required.

We can model this situation as a simple game({1,2,3},v) where v(N)=1, v(1)=v(1,2)=v(1,3)=1, and v(S)=0 for the remaining coalitions.

Unanimity games

- An important subclass of *simple* games is the class of unanimity games
- DEF Let $T \in 2^N \setminus \{\emptyset\}$. The *unanimity game* on T is defined as the TU-game (N, u_T) such that

$$u_T(S) = \begin{cases} 1 & \text{is } T \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

- Note that the class $\boldsymbol{G}^N$ of all n-person TU-games is a vector space (obvious what we mean for v+w and αv for $v, w \in G^N$.
- the dimension of the vector space G^N is 2^n-1
- $\{u_T \mid T \in 2^N \setminus \{\emptyset\}\}$ is an interesting basis for the vector space G^N.