Introduction to Game Theory and Applications

Stefano MORETTI and Fioravante PATRONE

LAMSADE (CNRS), Paris Dauphine
and
DIPTEM, University of Genoa

Paris, Telecom ParisTech, 2010

Summary

Potential games

- Exact potential games
- Ordinal potential games
- Network traffic
- Congestion games

More on oligopoly

Commons, interference, externalities: all made simple ;-)

Definition

A strategic form game $G = (X, Y, f, g)$ is said to be an (exact) potential game if:

There exists $P : X \times Y \to \mathbb{R}$ s.t.:
$\forall x_1, x_2 \in X, \forall y \in Y : \quad P(x_1, y) - P(x_2, y) = f(x_1, y) - f(x_2, y)$
$\forall x \in X, \forall y_1, y_2 \in Y : \quad P(x, y_1) - P(x, y_2) = g(x, y_1) - g(x, y_2)$

P is said to be a potential for G.

Results:
P is a potential iff $P + c$ is a potential ($c \in \mathbb{R}$)
P is a potential iff "integration" along all cycles is zero.

Coordination & dummy

(X, Y, P, P) is a *pure coordination game* (players have the same payoffs).

A game G is an (exact) potential game iff $G = G_c + G_d$, where:
- G_c is a pure coordination game
- G_d is a dummy (or pure externality) game: $f(x_1, y) = f(x_2, y)$ for all x_1, x_2, y

Example (already seen, Monday): the PD.

$I \backslash II$	L	R
T	0 0	1 1
B	1 1	2 2

$+$

$I \backslash II$	L	R
T	3 3	0 3
B	3 0	0 0

The importance of being dummy

We have seen the PD:

$I \backslash II$	L		R	
T	0	0	1	1
B	1	1	2	2

$+$

$I \backslash II$	L		R	
T	3	3	0	3
B	3	0	0	0

It is due to the "dummy" part of the decomposition the reason for the inefficiency result.

A leitmotiv that often occurs, especially in applications.

Nash equilibria

Given a potential game $G = (X, Y, f, g)$, it is *obvious from the definition* that $(\bar{x}, \bar{y})$ is a NE for G iff $(\bar{x}, \bar{y})$ is a NE for (X, Y, P, P).

Theorem (obvious):
If $(\bar{x}, \bar{y})$ maximizes P, then $(\bar{x}, \bar{y})$ is a NE.

Corollary:
A finite game with potential has a NE in pure strategies

Computationally interesting. It reduces the search for a NE to a search for maximum. Much easier.

Nash equilibria

Remark: there are NE that are not potential maximizers, so we are not sure to find all NE.

$I \backslash II$	L	R
T	1 1	0 0
B	0 0	0 0

Potential maximizers can be seen as a refinement of NE.

B.t.w. (B, R) is not a perfect (hence neither proper) NE.
NE is perfect if it is limit of (at least) one sequence of NE for appropriate restrictions to the use of "completely mixed" strategies.
A perfect Nash equilibrium is also called "trembling hand (Nash) equilibrium".

Ordinal potential games

Let us recall the definition of an exact potential game. We have (for player I):

$$\forall x_1, x_2, y : \quad P(x_1, y) - P(x_2, y) = f(x_1, y) - f(x_2, y) \;\; (1)$$

From this it follows:

$$\forall x_1, x_2, y : \quad P(x_1, y) > P(x_2, y) \iff f(x_1, y) > f(x_2, y) \;\; (2)$$

We cannot go back from (2) to (1). It is a weaker condition.
So, we speak of ordinal potential game. Notice that we can rewrite (2) using only preferences:

$$\forall x_1, x_2, y : \quad (x_1, y) \sqsupset (x_2, y) \iff (x_1, y) \succ_I (x_2, y) \;\; (2')$$
$$\forall x, y_1, y_2 : \quad (x, y_1) \sqsupset (x, y_2) \iff (x, y_1) \succ_{II} (x, y_2) \;\; (2')$$

Where $\sqsupset$, $\succ_I$ and $\succ_{II}$ are preferences induced on $X \times Y$...

Oligopoly game

Cournot model. Two mineral water "producers", that make simultaneous decisions on the quantity to produce.

Costs: $c_i(q) = cq$ (linear costs, with the same coefficient for both players) for $i = I, II$

Inverse demand function: $F :]0, +\infty[\to \mathbb{R}$, positive. $F(q)$ is the market clearing price, given the *produced quantity q*
Payoffs: $f(q_I, q_{II}) = F(q_I + q_{II})q_I - cq_I$ and similarly for II.

Ordinal potential game. A potential can be:
$P(q_I, q_{II}) = q_I q_{II}(F(q_I + q_{II}) - c)$.

The problem and the choices of the players

The very simple network is described pictorially in the file:
`GT_Telecom_ParisTech_2010_Wednesday_FP_network_1.pdf`

A couple of choices from players *I* and *II* are put in evidence in the file:
`GT_Telecom_ParisTech_2010_Wednesday_FP_network_2.pdf`

Let's build the strategic form

Let's fill the entries in the matrix:

$I \backslash II$	L_1		L_2	
L_1	$C_1^I +$	$D_1^{II} +$	$C_1^I + D_2^I$	$D_1^{II} + C_2^{II}$
L_2	$+$	$+$	$+ D_2^I$	$+ C_2^{II}$

I chooses L_1 and II chooses L_2

The complete game

The (bi)matrix of the game is:

$I \backslash II$	L_1	L_2
L_1	$C_1^I + D_1^I \quad D_1^{II} + C_1^{II}$	$C_1^I + D_2^I \quad D_1^{II} + C_2^{II}$
L_2	$C_2^I + D_1^I \quad D_2^{II} + C_1^{II}$	$C_2^I + D_2^I \quad D_2^{II} + C_2^{II}$

Split into coordination and dummy

Sum of two games:

$I \backslash II$	L_1	L_2
L_1	$C_1^I \quad C_1^{II}$	$C_1^I \quad C_2^{II}$
L_2	$C_2^I \quad C_1^{II}$	$C_2^I \quad C_2^{II}$

$I \backslash II$	L_1	L_2
L_1	$D_1^I \quad D_1^{II}$	$D_2^I \quad D_1^{II}$
L_2	$D_1^I \quad D_2^{II}$	$D_2^I \quad D_2^{II}$

The potential is:

$I \backslash II$	L_1	L_2
L_1	0	$C_2^{II} - C_1^{II}$
L_2	$C_2^I - C_1^I$	$C_2^I - C_1^I + C_2^{II} - C_1^{II}$

Remark: the game above on the left is not a pure coordination game! So, we don't have a split into coordination + dummy game. For this, see the next slide.

Split into coordination and dummy

Easy to get the "required" split. Just consider the game (X, Y, P, P), which is a coordination game. The dummy game is given by $(X, Y, f - P, g - P)$:

$I \backslash II$	L_1	L_2
L_1	$C_1^I + D_1^I \quad D_1^{II} + C_1^{II}$	$C_1^I + D_2^I \quad D_1^{II} + C_2^{II}$
L_2	$C_2^I + D_1^I \quad D_2^{II} + C_1^{II}$	$C_2^I + D_2^I \quad D_2^{II} + C_2^{II}$

$-$

$I \backslash II$	L_1	L_2
L_1	0	$C_2^{II} - C_1^{II}$
L_2	$C_2^I - C_1^I$	$C_2^I - C_1^I + C_2^{II} - C_1^{II}$

The difference of these two games, as can be checked, is a dummy game.

Useful conditions

Which are useful ingredients for a good soup, i.e. to get a potential game?

Payoffs from the choice of (x, y) are derived *additively* by the payoffs due to the choice of x (affecting both I and II) and the payoffs due to the choice of y (affecting...)

Useful conditions

If conditions of the previous slide are satisfied, we have a game like this:

$I \backslash II$	L		R	
T	$\phi_I(T) + \psi_{II}(L)$	$\phi_{II}(L) + \psi_I(T)$	$\phi_I(T) + \psi_{II}(R)$	$\phi_{II}(R) + \psi_I(T)$
B	$\phi_I(B) + \psi_{II}(L)$	$\phi_{II}(L) + \psi_I(B)$	$\phi_I(B) + \psi_{II}(R)$	$\phi_{II}(R) + \psi_I(B)$

Where:
- ϕ_I describes the costs for player I, due to the choice of strategy made by player I himself.
- ϕ_{II} describes the costs for player II, due to the choice of strategy made by player II himself.
- ψ_I describes the costs for player II, due to the choice of strategy made by player I (externality).
- ψ_{II} describes the costs for player I, due to the choice of strategy made by player II (externality).

Greedy & dummy

It is obvious, from the definition of the ϕ's and ψ's, that the game can be split as sum of a "greedy" game (the game of ϕ's) and a "dummy" game (the game of ψ's):

$I \backslash II$	L	R
T	$\phi_I(T)$ $\phi_{II}(L)$	$\phi_I(T)$ $\phi_{II}(R)$
B	$\phi_I(B)$ $\phi_{II}(L)$	$\phi_I(B)$ $\phi_{II}(R)$

$+$

$I \backslash II$	L	R
T	$\psi_{II}(L)$ $\psi_I(T)$	$\psi_{II}(R)$ $\psi_I(T)$
B	$\psi_{II}(L)$ $\psi_I(B)$	$\psi_{II}(R)$ $\psi_I(B)$

Of course (we did not lie, two slides ago), we can make also the decomposition into "pure coordination" + "dummy" game. See next slide.

Potential (pure coordination) & dummy

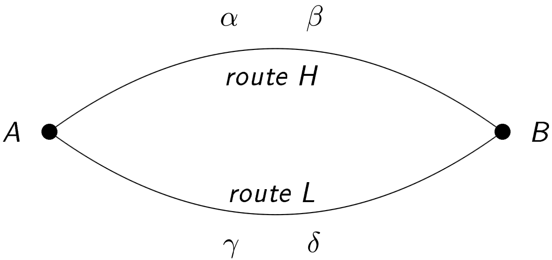
We decompose here the given game into the game (X, Y, P, P) and find the dummy game by difference (calculations not shown...). First we represent the potential (we write only P in the cells, not "P, P"):

$I \backslash II$	L	R
T	0	$\phi_{II}(R) - \phi_{II}(L)$
B	$\phi_I(B) - \phi_I(T)$	$\phi_I(B) - \phi_I(T) + \phi_{II}(R) - \phi_{II}(L)$

The "dummy" game is:

$I \backslash II$	L	R
T	$\phi_I(T) + \psi_{II}(L) \quad \phi_{II}(L) + \psi_I(T)$	$\phi_I(T) + \psi_{II}(R) - \phi_{II}(R) + \phi_{II}(L) \quad \phi_{II}(L) + \psi_I(T)$
B	$\phi_I(T) + \psi_{II}(L) \quad \phi_{II}(L) + \psi_I(B) - \phi_I(B) + \phi_I(T)$	$\phi_I(T) + \psi_{II}(R) - \phi_{II}(R) + \phi_{II}(L) \quad \phi_{II}(L) + \psi_I(B) - \phi_I(B) + \phi_I(T)$

Congestion game



- α is the cost for each player using the road if exactly one player uses the route H
- $\alpha + \beta$ is the cost for each player using the road if exactly two players uses the route H

Similarly for route L. We assume that $\alpha, \beta, \gamma, \delta > 0$.

Congestion games

Easy to check that it is a potential game.

Namely, in strategic form we get:

$I \backslash II$	H	L
H	$\alpha+\beta \quad \alpha+\beta$	$\alpha \quad \gamma$
L	$\gamma \quad \alpha$	$\gamma+\delta \quad \gamma+\delta$

More than this: all congestion games are potential games, and vice-versa. Result by Rosenthal (1973).

Cournot, the simplest model

Costs: $c_i(q) = cq$ (linear costs, with the same coefficient for both players) for $i = I, II$

Inverse demand function: $F : [0, +\infty[\to \mathbb{R}$. $F(q) = \max\{a - q, 0\}$. We assume $a > c$ (if not, producing is not convenient).

Payoffs: $f(q_I, q_{II}) = F(q_I + q_{II})q_I - cq_I$ and similarly for II.

Easy to find the (unique) NE: $(\frac{a-c}{3}, \frac{a-c}{3})$. May be reached by iterated elimination of strongly dominated strategies.

Inefficient! Payoff is $\frac{(a-c)^2}{9}$ for each firm. See next slide.

Cournot, collusion

Players could reach a (binding) agreement, i.e. could create a cartel.

They could decide to find the monopoly optimal quantity to produce (it is $\frac{a-c}{2}$), produce each half of it and thus share evenly the oligopoly payoff which is $\frac{(a-c)^2}{4}$.

In such a way each firm would get $\frac{(a-c)^2}{8}$, which is better than what gets in oligopoly. This proves the inefficiency mentioned in the previous slide.

Which is the game being played?

So, cartels are nice? Why so the anti-cartel (anti-trust) legislation and regulations?

Because there is a larger game, in which consumers play also. Notice that from oligopoly to carte the quantity produced diminishes (from $\frac{2}{3}(a-c)$ to $\frac{1}{2}(a-c)$) and prices grow up (from $\frac{a+2c}{3}$ to $\frac{a+c}{2}$). So, consumers won't be happy.

General lesson learned: which is the true game being played?

Stackelberg model

Assume that one firm is the "leader" and the other is a "follower". To be understood essentially as having more (respectively: less) market power.
So, the leader firm decides the quantity and then, having observed its decision, the follower makes its optimal choice.

A typical game in extensive form with perfect information.
Easy to solve by backward induction (the structural data are the same as in Cournot's model).

Result: a SPE which is $(\frac{a-c}{2}, \frac{a-c}{4})$ (I is the leader). The profit is $(\frac{(a-c)^2}{8}, \frac{(a-c)^2}{16})$. The resulting market equilibrium price is $\frac{a+3c}{4}$.

We leave to the reader the comparisons with the other models.

Bertrand oligopoly

Often competition among firms is on prices, not on quantity produced.

The Bertrand model.
We use the same structural data. In particular, the quantity sold depends on the minimum of the prices settled by the firms: the demand function is $q = \max\{a - p, 0\}$, where $p = \min\{p_1, p_2\}$.

Assumptions on the dependence of quantity sold from prices: the firm posting the smallest price gets all of the market; in case they post the same price, they share evenly the market.

Bertrand oligopoly

It can be proved that there is a unique NE, in which each firm settles the price equal to the unit cost of production (= marginal costs, since we have assumed constant returns to scale).

Surprising results? Well, the assumptions made are really of a cut-throat behavior of the consumers.
Or, assuming a bit of differentiation between the products of the two firms, less dire results appear.

Simple, simple model

$N = \{1, \ldots, n\}$ is the set of players
$X_i = [0, +\infty[$ is the strategy space for every player $i \in N$
$f_i(x_1, \ldots, x_n) = \phi(x_i) - \psi(x_{-i})$ is the payoff function for player $i \in N$:
- direct earnings, described by ϕ
- minus externalities, coming from the remaining players, described by ψ

Let's make it as simple as possible!
$\phi(s) = K^2 - (s - K)^2 = s(2K - s)$ (earnings grow for $s \leq K$, then start to decline)
$\psi(x_{-i}) = \nu(\sum_{j \neq i} x_j)$ (not important the identities of players producing externalities: only the cumulative effect is important)

Even simpler!!
$\nu(t) = \alpha \cdot t$ (α is a real positive number).

Looking for Nash equilibria

Given $\bar{x}_{-i}$, $f_i(x_i, x_{-i}) = K^2 - (x_i - K)^2 - \alpha \cdot (\sum_{j \neq i} \bar{x}_j)$.
Clearly, $R_i(\bar{x}_{-i}) = K$ for any $\bar{x}_{-i}$: best reply is constant (K is a dominating strategy!).

So, obviously, $(K, \ldots, K)$ is a Nash equilibrium (in dominant strategies).

Payoff?
$f_i(K, \ldots, K) = K^2 - (K - K)^2 - \alpha \cdot (n-1)K = K^2 - \alpha \cdot (n-1)K$.

Looking for social optimum

Due to the symmetry of the problem, let's look at what can be achieved, provided that all of the players use the same strategy z.
The payoff for player i is:
$f_i(z, \dots, z) = K^2 - (z - K)^2 - \alpha \cdot (n-1)z.$

Since it is a (strictly concave) second degree polynomial, we find its maximum by finding its critical point, that is solving:
$\frac{f_i(z,\dots,z)}{dz} = -2(z - K) - \alpha \cdot (n-1) = 0$
From which we get: $z = K - \frac{\alpha}{2} \cdot (n-1).$

Since $K \neq K - \frac{\alpha}{2} \cdot (n-1)$, the Nash equilibrium does not coincide with the unique (symmetric) socially optimal choice, *ergo* the Nash equilibrium is inefficient (or see directly that the "social" profit for a player is $K^2 - \alpha \cdot (n-1)K + \frac{\alpha^2}{4} \cdot (n-1)^2$, better than in NE).

Comments

The difference between the Nash equilibrium and the social optimum is given by $\frac{\alpha}{2} \cdot (n-1)$.

So, how far is Nash equilibrium from the social optimal choice? This is due to the factors α and n: even with a small α, the resulting externalities can have a significant effect if there are many players.

This is the basic phenomenon that we see in the "tragedy of the commons"-like situations. Or in oligopoly (if oligopolists are many, profits converge to the competitive profits, so the oligopolists see their monopolistic-like extra earnings wiped out). Or in cases in which you have interference...