Introduction to Game Theory and Applications

Stefano MORETTI and Fioravante PATRONE

LAMSADE (CNRS), Paris Dauphine
and
DIPTEM, University of Genoa

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Summary

Games in extensive form

Repeated games

Matching pennies revisited

Let us redefine the rules of the game.
- first player I chooses, T or B
- then, II chooses: L or R

Is it ok? Is it the same game?

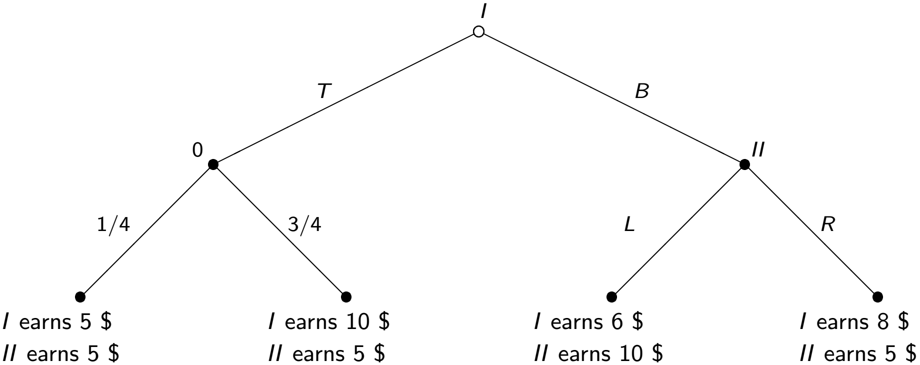
It depends.

Essential is *not* the chronological (physical) time, but the information that II has when he must decide.
So, II can see (know) the choice of I before deciding?

Trees

We have introduced two important aspects:
- the dynamic structure of the interaction
- the role of info on past events (on the *history*)

To represent them it is appropriate a tree structure.



Information sets

A couple of examples.

Figure 1: a simple case of sequential decision making. First chooses *I*. Then, being informed of the choice made by *I*, *II* makes his choice.

Figure 2: a "trick" to take note of what *II* does not know about the past (the history).

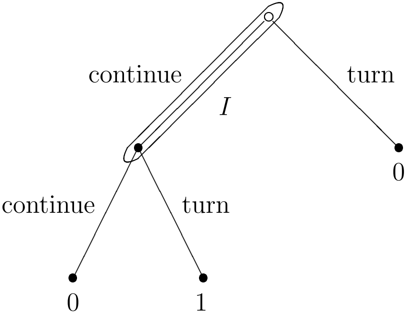
Two nodes are connected with a dashed line. The meaning is that a player will know that has to make a choice knowing that:
- he knows he is in one of these nodes
- but he does not know in which.

These set of nodes connected with dashed lines are called "information sets".

More on information sets

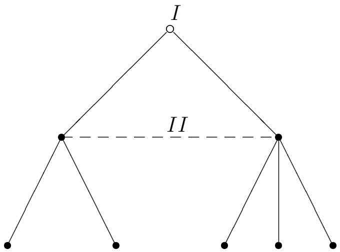
A condition that information sets must satisfy (according to the definition given by Kuhn in 1953) is that any path from the root to a terminal node can contain at most one node of an information set.

This means that a game like this (Isbell game) is not allowed:



More on information sets

Another condition that information sets must satisfy is that the number of branches leaving the nodes of an information set must be equal for all of the nodes.



If this condition were violated, we could say that:
- either the player (to which the information node refers) is unable to make a choice;
- or that he does not ignore in which node (of the information set) is, since he has (at least for a couple of nodes) a different number of choices available, depending on the node where he is.

Chance moves

It often happens, in “parlor” games, that there are “chance moves”.
- chess: who will play with white pieces?
- cards: mix well cards, that is choose randomly one of the permutations of the card’s deck
- backgammon: every time, dice will decide how many movements the player can do

We have already used it in the figure in slide 3. We use player “0” as the “chance” player. And we have a probability distribution on the branches exiting from the chance node.

Strategy

Consider the very simple game depicted in figure 3 (matches' game; f stands for the initial of the Italian word for matches: "fiammiferi"):
Player I must begin choosing between $1f$ or $2f$. But there is nothing that obliges him to think locally.

He knows that he could be called to play again. So, before the game starts, he can decide his strategy.
It means, choose among: $1f\ 1f$, $1f\ 2f$, $2f\ 1f$, $2f\ 2f$.
Similarly, II can choose among: $1f\ 1f$, $1f\ 2f$, $2f\ 1f$, $2f\ 2f$ (symbols are the same, but their meaning is different).

So, we have a game in strategic form. Payoffs? Follow the path! And maybe expected payoffs, in case of chance moves.

Nothing new...

Every game in extensive form can be converted into a game in strategic form, using the (natural) idea of strategy for a player that we have seen.
Remark: the assumption on the intelligence of the players is essential!

So, we can say that the strategic form is, somehow, fundamental (see vN '28). At least, for the non-cooperative case.

We can use Nash equilibrium also a solution for a game in extensive form.

It seems that everything is so easy...

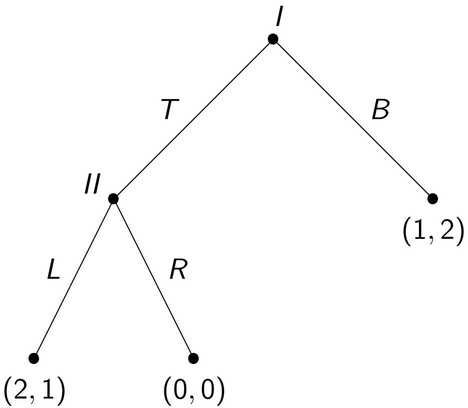
Backward induction

Consider the “matches” game. We can easily “solve” it.

Look at the “penultimate” nodes. There the choice is easy, it is a single DM that has all of the power to enforce the outcome he prefers.
Having done this, look at the pre-penultimate nodes. Taking into account the choices that will be made by the player who follows, the choice for the player at the pre-penultimate node becomes obvious too.
And so on, till we reach the root of the tree.
The method we followed is called backward induction and works for games with perfect information (i.e.: information sets are all singletons).
Well, we get a strategy profile that has good chances to be considered a solution! Actually, it can be proved that it is a Nash equilibrium.

A small problem

Let's see a game, very small:



Backward induction gives (T, L). But the strategic form has two Nash equilibria: (T, L) and (B, R)!
And so?

Subgame Perfect Equilibrium

The key point is that the strategy profile determined by backward induction is more than a NE. It is a SPE. That is, it is not only a NE, but it is also a NE *for all of the subgames*.
What are subgames? For games with perfect information $\sim$ subtrees. More interesting the general case, but the idea is obvious: we want a subtree that can be seen sensibly as a game (subgame of the given game).

Games with perfect information: there is coincidence between SPE and strategy profiles found by backward induction.

SPE can be defined for any game in extensive form. It is the first example of refinement of NE.

A final comment

Notice that a game in strategic form can be described as a game in extensive form. Jut put player I at the root node. Branches leaving the root will correspond to the strategies of player I.

Then, at each of the nodes at the end of these branches one attaches as many branches as strategies for player II; moreover, all of these nodes "belonging" to player II have to be included in the same information set.

Label final nodes with the payoffs in the obvious way, and you are done.

Remark: of course, the order of players could be inverted.

Exercise: describe this transformation for a game in strategic form with an arbitrary (but finite) set of players N.

A preliminary case

What happens if a game is repeated?
Let's start with a simple example. A (small) game played twice.
See pages attached.

Notice that, from the extensive form, we see that the results of the first stage are *observed by both players* before deciding what to do in the second stage.

Interesting facts:
- just one SPE
- 16 NE, all inducing (however) exactly the same actions: differences invisible to an external observer

Apparently, from repetition nothing interesting comes out.

The most extreme case: infinite stages

PD, infinitely repeated.

Remark: discounted payoffs (many things should be said...)

Consider the following strategy for I:
- I plays T in the first stage
- I plays T also in all of the following stages, unless he observes II playing R. In such a case, I will play B forever

Assume that II plays the twin strategy.
This couple of strategies is a NE, provided that the discount factor is not too small.

Deviations do not pay, unless...

Let us see whether a deviation pays:
- no deviation: payoff for I is: $\sum_{n=0}^{\infty} 3\delta^n = \frac{3}{1-\delta}$
- I deviates at the 1^{st} stage (for other stages, calculations are essentially the same): $\leq 4 + \sum_{n=1}^{\infty} 2\delta^n = 2 + \frac{2}{1-\delta}$

So, to deviate does not pay if $\delta \geq 1/2$

Obvious that for a very impatient player (has to "honor" tomorrow a debt with mafia...) deviation pays

But very interesting that for not so impatient players there is the possibility of getting (in equilibrium!!) an efficient outcome for the infinitely repeated game. Striking contrast with the 2 stages case (and in general with the n stages case).

Terre di mezzo

We have seen two radically different results.

Is there anything "in the middle"?

The answer is: yes!

Examples:

- random termination (at the end of each stage, the probability that the game continues is 0.99)

- some doubts about the rationality of the other player (TIT FOR TAT machine): see Kreps *et al.*, 1982.