GAME THEORY

NON-COOPERATIVE THEORY
- Games in extendive form (tree games)
- Games in strategic form (normal form)
  - Dominant strategies
  - Nash eq. (NE)
  - Subgame perfect NE
  - NE & refinements

COORDINATIONAL THEORY
- Games in c.f.f. (TU-games or coalitional games)
  - Core
  - Shapley value
  - Nucleolus
  - $\tau$-value
  - PMAS

COOPERATIVE THEORY
- Bargaining games
  - Nash sol.
  - Kalai-Smorodinsky
- NTU-games
  - CORE
  - NTU-value
  - Compromise value

No binding agreements
No side payments
Q: Optimal behaviour in conflict situations

binding agreements
side payments are possible (sometimes)
Q: Reasonable (cost, reward)-sharing
How to share $v(N)$...

- The Core of a game can be used to exclude those allocations which are *not stable*.
- But the core of a game can be a bit “extreme” (see for instance the glove game).
- Sometimes the core is *empty* (see for example the game with pirates).
- And if it is not empty, there can be many allocations in the core (*which is the best?*)
An axiomatic approach (Shapley (1953))

- Similar to the approach of Nash in bargaining: which properties an allocation method should satisfy in order to divide $v(N)$ in a reasonable way?
- Given a subset $C$ of $G^N$ (class of all TU-games with $N$ as the set of players) a \textit{(point map) solution} on $C$ is a map $\Phi: C \rightarrow \mathbb{R}^N$.
- For a solution $\Phi$ we shall be interested in various properties...
Symmetry

**PROPERTY 1 (SYM)** For all games \( v \in \mathbf{G}^N \),

If \( v(S \cup \{i\}) = v(S \cup \{j\}) \) for all \( S \in 2^N \) s.t. \( i,j \in N \setminus S \),

then \( \Phi_i(v) = \Phi_j(v) \).

**EXAMPLE**

Consider a TU-game \( \{1,2,3\},v \) s.t. \( v(1) = v(2) = v(3) = 0 \), \( v(1, 2) = v(1, 3) = 4 \), \( v(2, 3) = 6 \), \( v(1, 2, 3) = 20 \).

Players 2 and 3 are symmetric. In fact:

\( v(\emptyset \cup \{2\}) = v(\emptyset \cup \{3\}) = 0 \) and \( v(\{1\} \cup \{2\}) = v(\{1\} \cup \{3\}) = 4 \)

If \( \Phi \) satisfies \( \text{SYM} \), then \( \Phi_2(v) = \Phi_3(v) \)
Efficiency

**PROPERTY 2 (EFF)** For all games \(v \in G^N\),
\[\sum_{i \in N} \Phi_i(v) = v(N),\] i.e., \(\Phi(v)\) is a pre-imputation.

Null Player Property

**DEF.** Given a game \(v \in G^N\), a player \(i \in N\) s.t.
\[v(S \cup i) = v(S)\] for all \(S \in 2^N\) will be said to be a null player.

**PROPERTY 3 (NPP)** For all games \(v \in G^N\), \(\Phi_i(v) = 0\) if \(i\) is a null player.

**EXAMPLE** Consider a TU-game \((\{1,2,3\}, v)\) such that
\[v(1) = 0, \ v(2) = v(3) = 2, v(1, 2) = v(1, 3) = 2, v(2, 3) = 6, v(1, 2, 3) = 6.\] Player 1 is a null player. Then \(\Phi_1(v) = 0\)
**EXAMPLE** Consider a TU-game \((\{1,2,3\},v)\) such that 
\[v(1) = 0, \; v(2) = v(3) = 2, \; v(1, 2) = v(1, 3) = 2, \; v(2, 3) = 6, \; v(1, 2, 3) = 6.\] On this particular example, if \(\Phi\) satisfies NPP, SYM and EFF we have that

\[\Phi_1(v) = 0 \text{ by NPP} \]
\[\Phi_2(v) = \Phi_3(v) \text{ by SYM} \]
\[\Phi_1(v) + \Phi_2(v) + \Phi_3(v) = 6 \text{ by EFF} \]

So \(\Phi = (0, 3, 3)\)

But our goal is to characterize \(\Phi\) on \(G^N\).

One more property is needed.
Additivity

**PROPERTY 2 (ADD)** Given \( v, w \in G^N \),

\[ \Phi(v) + \Phi(w) = \Phi(v + w). \]

**EXAMPLE** Consider two TU-games \( v \) and \( w \) on \( N = \{1, 2, 3\} \):

\[
\begin{align*}
  v(1) &= 3 \\
  v(2) &= 4 \\
  v(3) &= 1 \\
  v(1, 2) &= 8 \\
  v(1, 3) &= 4 \\
  v(2, 3) &= 6 \\
  v(1, 2, 3) &= 10 \\
\end{align*}
\]

\[
\begin{align*}
  w(1) &= 1 \\
  w(2) &= 0 \\
  w(3) &= 1 \\
  w(1, 2) &= 2 \\
  w(1, 3) &= 2 \\
  w(2, 3) &= 3 \\
  w(1, 2, 3) &= 4 \\
\end{align*}
\]

\[
\begin{align*}
  v + w(1) &= 4 \\
  v + w(2) &= 4 \\
  v + w(3) &= 2 \\
  v + w(1, 2) &= 10 \\
  v + w(1, 3) &= 6 \\
  v + w(2, 3) &= 9 \\
  v + w(1, 2, 3) &= 14 \\
\end{align*}
\]
**Theorem 1** (Shapley 1953)

There is a unique point map solution \( \phi \) defined on \( G^N \) that satisfies EFF, SYM, NPP, ADD. Moreover, for any \( i \in N \) we have that

\[
\phi_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi} m_i^\sigma(v)
\]

Here \( \Pi \) is the set of all permutations \( \sigma: N \rightarrow N \) of \( N \), while \( m_i^\sigma(v) \) is the marginal contribution of player \( i \) according to the permutation \( \sigma \), which is defined as:

\[
v(\{\sigma(1), \sigma(2), \ldots, \sigma(j)\}) - v(\{\sigma(1), \sigma(2), \ldots, \sigma(j-1)\})
\]

where \( j \) is the unique element of \( N \) s.t. \( i = \sigma(j) \).
Probabilistic interpretation: (the “room parable”)

- Players gather one by one in a room to create the “grand coalition”, and each one who enters gets his marginal contribution.
- Assuming that all the different orders in which they enter are equiprobable, the Shapley value gives to each player her/his expected payoff.

Example

(N,v) such that
N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2)=3,
v(1,3)=1,
v(2,3)=4,
v(1,2,3)=5.

\[
\begin{array}{cccc}
\text{Permutation} & 1 & 2 & 3 \\
1,2,3 & 0 & 3 & 2 \\
1,3,2 & 0 & 4 & 1 \\
2,1,3 & 0 & 3 & 2 \\
2,3,1 & 1 & 3 & 1 \\
3,2,1 & 1 & 4 & 0 \\
3,1,2 & 1 & 4 & 0 \\
\text{Sum} & 3 & 21 & 6 \\
\phi(v) & 3/6 & 21/6 & 6/6 \\
\end{array}
\]
Example

\((N,v)\) such that \(N=\{1,2,3\}\),
\(v(1)=v(3)=0,\)
\(v(2)=3,\)
\(v(1,2)=3, v(1,3)=1\)
\(v(2,3)=4\)
\(v(1,2,3)=5.\)

\(\phi(v)=(0.5,3.5,1)\)
Exercise

Calculate the Shapley value of the TU-game $(N, v)$ such that $N = \{1, 2, 3\}$,
$v(1) = 3$
$v(2) = 4$
$v(3) = 1$
$v(1, 2) = 8$
$v(1, 3) = 4$
$v(2, 3) = 6$
$v(1, 2, 3) = 10$

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<tr>
<th>Permutation</th>
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<td>3, 1, 2</td>
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<tr>
<td>$\phi(v)$</td>
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</table>
Example
(Glove game with $L=\{1,2\}$, $R=\{3\}$)
$v(1,3)=v(2,3)=v(1,2,3)=1, \quad v(S)=0$ otherwise

\[
\begin{array}{c|c|c|c}
\text{Permutation} & 1 & 2 & 3 \\
\hline
1,2,3 & 0 & 0 & 1 \\
1,3,2 & 0 & 0 & 1 \\
2,1,3 & 0 & 0 & 1 \\
2,3,1 & 0 & 0 & 1 \\
3,2,1 & 0 & 1 & 0 \\
3,1,2 & 1 & 0 & 0 \\
\hline
\text{Sum} & 1 & 1 & 4 \\
\hline
\phi(v) & 1/6 & 1/6 & 4/6 \\
\end{array}
\]
Unanimity games (1)

**DEF** Let \( T \in 2^N \setminus \{ \emptyset \} \). The *unanimity game* on \( T \) is defined as the TU-game \((N,u_T)\) such that

\[
\begin{cases}
1 \text{ if } T \subseteq S \\
0 \text{ otherwise}
\end{cases}
\]

\( u_T(S) = \)

Note that the class \( G^N \) of all n-person TU-games is a vector space (obvious what we mean for \( v+w \) and \( \alpha v \) for \( v,w \in G^N \) and \( \alpha \in \text{IR} \)).

- the dimension of the vector space \( G^N \) is \( 2^n - 1 \), where \( n = |N| \).

\( \{ u_T | T \in 2^N \setminus \{ \emptyset \} \} \) is an interesting basis for the vector space \( G^N \).
Every coalitional game \((N, v)\) can be written as a linear combination of unanimity games in a unique way, i.e., \(v = \sum_{S \in 2^N} \lambda_S(v)u_S\).

The coefficients \(\lambda_S(v)\), for each \(S \in 2^N\), are called unanimity coefficients of the game \((N, v)\) and are given by the formula: \(\lambda_S(v) = \sum_{T \in 2^s} (-1)^{s-t} v(T)\).
EXAMPLE Unanimity coefficients of ([1,2,3],v)

\[ v(1) = 3 \quad \lambda_1(v) = 3 \]

\[ v(2) = 4 \quad \lambda_2(v) = 4 \]

\[ v(3) = 1 \quad \lambda_3(v) = 1 \]

\[ v(1, 2) = 8 \quad \lambda_{\{1,2\}}(v) = -3 - 4 + 8 = 1 \]

\[ v(1, 3) = 4 \quad \lambda_{\{1,3\}}(v) = -3 - 1 + 4 = 0 \]

\[ v(2, 3) = 6 \quad \lambda_{\{2,3\}}(v) = -4 - 1 + 6 = 1 \]

\[ v(1, 2, 3) = 10 \quad \lambda_{\{1,2,3\}}(v) = -3 - 4 - 1 + 8 + 4 + 6 - 10 = 0 \]

\[ v = 3u_{\{1\}}(v) + 4u_{\{2\}}(v) + u_{\{3\}}(v) + u_{\{1,2\}}(v) + u_{\{2,3\}}(v) \]
Sketch of the Proof of Theorem 1

- Shapley value satisfies EFF, SYM, NPP, ADD (“easy” to prove).
- Properties EFF, SYM, NPP determine $\phi$ on the class of all games $\alpha v$, with $v$ a unanimity game and $\alpha \in \mathbb{IR}$.
  - Let $S \in 2^N$. The Shapley value of the unanimity game $(N, u_S)$ is given by
    $$\phi_i(\alpha u_S) = \begin{cases} 
    \alpha/|S| & \text{if } i \in S \\
    0 & \text{otherwise}
    \end{cases}$$
- Since the class of unanimity games is a basis for the vector space, ADD allows to extend $\phi$ in a unique way to $G^N$. 
An alternative formulation

- Let \( m_{\sigma_i}(v) = v(\{\sigma'(1), \sigma'(2), \ldots, \sigma'(j)\}) - v(\{\sigma'(1), \sigma'(2), \ldots, \sigma'(j-1)\}) \), where \( j \) is the unique element of \( \mathbb{N} \) s.t. \( i = \sigma'(j) \).
- Let \( S = \{\sigma'(1), \sigma'(2), \ldots, \sigma'(j)\} \).
- **Q**: How many other orderings \( \sigma \in \Pi \) do we have in which \( \{\sigma(1), \sigma(2), \ldots, \sigma(j)\} = S \) and \( i = \sigma(j) \)?
- **A**: they are precisely \((|S|-1)! \times (|N|-|S|)!\)
- Where \((|S|-1)!\) is the number of orderings of \( S \setminus \{i\} \) and \((|N|-|S|)!\) is the number of orderings of \( N \setminus S \).
- We can rewrite the formula of the Shapley value as the following:

\[
\phi_i(v) = \sum_{S \in 2^N : i \in S} \frac{(s-1)! (n-s)!}{n!} (v(S) - v(S \setminus \{i\}))
\]
Alternative properties

- **PROPERTY 5** (Anonymity, ANON) Let \((N, v) \in G^N\), \(\sigma: N \to N\) be a permutation. Then, \(\Phi_{\sigma(i)}(\sigma v) = \Phi_i(v)\) for all \(i \in N\).
  
  Here \(\sigma v\) is the game defined by: \(\sigma v(S) = v(\sigma(S))\), for all \(S \in 2^N\).

  Interpretation: The meaning of ANON is that whatever a player gets via \(\Phi\) should depend only on the structure of the game \(v\), not on his “name”, i.e., the way in which he is labelled.

- **DEF** (Dummy Player) Given a game \((N, v)\), a player \(i \in N\) s.t. \(v(S \cup \{i\}) = v(S) + v(i)\) for all \(S \in 2^N\) will be said to be a dummy player.

- **PROPERTY 6** (Dummy Player Property, DPP) Let \(v \in G^N\). If \(i \in N\) is a dummy player, then \(\Phi_i(v) = v(i)\).

**NB**: often NPP and SYM are replaced by DPP and ANON, respectively.
Characterization on a subclass

- Shapley and Shubik (1954) proposed to use the Shapley value as a power index,

- ADD property does not impose any restriction on a solution map defined on the class of simple games $S^N$, which is the class of games such that $v(S) \in \{0,1\}$ (often is added the requirement that $v(N) = 1$).

- Therefore, the classical conditions are not enough to characterize the Shapley–Shubik value on $S^N$.

- We need a condition that resembles ADD and can substitute it to get a characterization of the Shapley–Shubik (Dubey (1975)) index on $S^N$:

**PROPERTY 7** (Transfer, TRNSF) For any $v,w \in S(N)$, it holds:

$$
\Phi(v \lor w) + \Phi(v \land w) = \Phi(v) + \Phi(w).
$$

Here $v \lor w$ is defined as $(v \lor w)(S) = (v(S) \lor w(S)) = \max\{v(S),w(S)\}$, and $v \land w$ is defined as $(v \land w)(S) = (v(S) \land w(S)) = \min\{v(S),w(S)\}$,
**Example** Two TU-games $v$ and $w$ on $N=\{1,2,3\}$

- $v(1) = 0$
- $v(2) = 1$
- $v(3) = 0$
- $v(1, 2) = 1$
- $v(1, 3) = 1$
- $v(2, 3) = 0$
- $v(1, 2, 3) = 1$

- $w(1) = 1$
- $w(2) = 0$
- $w(3) = 0$
- $w(1, 2) = 1$
- $w(1, 3) = 0$
- $w(2, 3) = 1$
- $w(1, 2, 3) = 1$

- $v \land w(1) = 0$
- $v \land w(2) = 0$
- $v \land w(3) = 0$
- $v \land w(1, 2) = 1$
- $v \land w(1, 3) = 0$
- $v \land w(2, 3) = 0$
- $v \land w(1, 2, 3) = 1$

- $v \lor w(1) = 1$
- $v \lor w(2) = 1$
- $v \lor w(3) = 0$
- $v \lor w(1, 2) = 1$
- $v \lor w(1, 3) = 1$
- $v \lor w(2, 3) = 1$
- $v \lor w(1, 2, 3) = 1$
Reformulations

Other axiomatic approaches have been provided for the Shapley value, of which we shall briefly describe those by Young and by Hart and Mas-Colell.

**PROPERTY 8** (Marginalism, MARG) A map \( \Psi : \mathcal{G}^N \rightarrow \mathbb{IR}^N \) satisfies MARG if, given \( v, w \in \mathcal{G}^N \), for any player \( i \in N \) s.t.

\[
v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S) \quad \text{for each} \quad S \in 2^N,\]

the following is true:

\[
\Psi_i(v) = \Psi_i(w).
\]

**Theorem 2** (Young 1988) There is a unique map \( \Psi \) defined on \( \mathcal{G}(N) \) that satisfies EFF, SYM, and MARG. Such a \( \Psi \) coincides with the Shapley value.
**Example**  Two TU-games v and w on N={1,2,3}

\[
\begin{align*}
\text{v(1)} &= 3 \\
\text{v(2)} &= 4 \\
\text{v(3)} &= 1 \\
\text{v(1,2)} &= 8 \\
\text{v(1,3)} &= 4 \\
\text{v(2,3)} &= 6 \\
\text{v(1,2,3)} &= 10 \\
\text{w(1)} &= 2 \\
\text{w(2)} &= 3 \\
\text{w(3)} &= 1 \\
\text{w(1,2)} &= 2 \\
\text{w(1,3)} &= 3 \\
\text{w(2,3)} &= 5 \\
\text{w(1,2,3)} &= 4
\end{align*}
\]

\[
\begin{align*}
\text{w(∅∪{3}) - w(∅)} &= v(∅∪{3}) - v(∅) = 1 \\
\text{w({1}∪{3}) - w({1})} &= v({1}∪{3}) - v({1}) = 1 \\
\text{w({2}∪{3}) - w(∅)} &= v({2}∪{3}) - v(∅) = 1 \\
\text{w({1,2}∪{3}) - w({1,2})} &= v({1,2}∪{3}) - v({1,2}) = 1
\end{align*}
\]

\[\Psi_3(v) = \Psi_3(w).\]
A quite different approach was pursued by Hart and Mas-Colell (1987).

To each game \((N, v)\) one can associate a real number \(P(N,v)\) (or, simply, \(P(v)\)), its *potential*.

The “partial derivative” of \(P\) is defined as

\[
D^i(P)(N, v) = P(N,v) - P(N \setminus \{i\}, v|_{N \setminus \{i\}})
\]

**Theorem 3** (Hart and Mas-Colell 1987) There is a unique map \(P\), defined on the set of all finite games, that satisfies:

1) \(P(\emptyset, v_0) = 0\),

2) \(\sum_{i \in N} D^iP(N,v) = v(N)\).

Moreover, \(D^i(P)(N, v) = \phi_i(v)\). [\(\phi(v)\) is the Shapley value of \(v\)]
there are formulas for the calculation of the potential.

For example, $P(N,v) = \sum_{S \in 2^N} \lambda_S / |S| \ (Harsanyi \ dividends)$

**Example**

- $v(1) = 3$
- $v(2) = 4$
- $v(3) = 1$
- $v(1, 2) = 8$
- $v(1, 3) = 4$
- $v(2, 3) = 6$
- $v(1, 2, 3) = 10$

<table>
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<tr>
<th>$\lambda_1(v)$</th>
<th>$\lambda_2(v)$</th>
<th>$\lambda_3(v)$</th>
<th>$\lambda_{{1,2}}(v)$</th>
<th>$\lambda_{{1,3}}(v)$</th>
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<th>$\lambda_{{1,2,3}}(v)$</th>
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<td>3</td>
<td>4</td>
<td>1</td>
<td>1</td>
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</table>

$P(\{1,2,3\},v) = 3 + 4 + 1 + 1/2 + 1/2 = 9$

$P(\{1,2\},v_{|\{1,2\}}) = 3 + 4 + 1/2 = 15/2$

$P(\{1,3\},v_{|\{1,3\}}) = 3 + 1 = 4$

$P(\{2,3\},v_{|\{2,3\}}) = 4 + 1 + 1/2 = 11/2$

$\phi_1(v) = P(\{1,2,3\},v) - P(\{2,3\},v_{|\{2,3\}}) = 9 - 11/2 = 7/2$

$\phi_2(v) = P(\{1,2,3\},v) - P(\{1,3\},v_{|\{2,3\}}) = 9 - 4 = 5$

$\phi_3(v) = P(\{1,2,3\},v) - P(\{1,2\},v_{|\{2,3\}}) = 9 - 15/2 = 3/2$
Communication networks

Networks ➔ several interpretations:

- A way to describe games in extensive form
- Physical connections between individuals, companies, cities...
- Cooperation or communication restrictions between players
  - communications can be described as undirected or directed graphs, hypergraphs, partitions.
Communication networks as undirected graphs:

- An *undirected graph* is a pair \((N,L)\) where
- \(N\) is a set of *vertices* (later, *agents* or *players*)
- \(L=\{ \{i,j\} \mid \{i,j\} \subseteq N, i \neq j \}\) is the set of *edges* (bilateral communication links)

A communication graph \((N,L)\) should be interpreted as a way to model restricted cooperation:

- Players can cooperate with each other if they are connected (*directly*, or *indirectly* via a path)
- Indirect communication between two players requires the cooperation of players on a connecting path.
Example

Consider the undirected graph \((N,L)\) with \(N=\{1,2,3,4,5,6,7\}\) and \(L=\{\{1,2\}, \{2,6\}, \{5,6\}, \{1,5\}, \{3,7\}, \{4,7\}\}\)

Some notations:

\[ L_2 = \{\{1,2\}, \{2,6\}\} \quad \text{N}\backslash\text{L}=\{\{1,2,5,6\}, \{3,4,7\}\} \quad \text{set of components} \]

\[ L_{-2} = \{\{5,6\}, \{1,5\}, \{3,7\}, \{4,7\}\} \quad \text{N}\backslash\text{L}_{-2}=\{\{1,5,6\}, \{3,4,7\}, \{2\}\} \]

\[ N(\{\{1,2\}, \{2,6\}, \{3,7\}\})=\{1,2,6,3,7\} \]
Communication within hypergraphs

- A hypergraph is a pair \((N, C)\) with \(N\) the player set and \(C\) a family of subsets of \(N\).
- An element \(H \in C\) is called a conference.
- Interpretation: communication between players in a hypergraph can only take place within a conference.

Example

Consider the hypergraph \((N,C)\) with \(N=\{1,2,3,4,5,6\}\) and \(C=\{\{1,3\}, \{3,4,5\}, \{2,5,6\}\}\)

Some notations:

- A path from 1 to 2: \((1,\{1,3\},3,\{3,4,5\},4,\{3,4,5\},5,\{2,5,6\},2)\)
- \(N \setminus C=\{N\}\) set of components
- If \(R=\{1,2,3,4,5\}\) then \(R \setminus C=\{\{1,3,4,5\},\{2\}\}\)
Communication within cooperation structure

- A *cooperation structure* is a pair \((N,B)\) with \(N\) the player set and \(B\) a partition of the player set \(N\).
- **Interpretation**: communication between players in a hypergraph can only take place between any subset of an element of the cooperation structure \(\rightarrow\text{Coalition structure}\) (Aumann and Dréze (1974), Myerson (1980), Owen (1977)).
Cooperative games with restricted communication

- A cooperative game describes a situation in which all players can freely communicate with each other.

- Drop this assumption and assume that communication between players is restricted to a set of communication possibilities between players.

- $L=\{ \{i,j\} \mid \{i,j\}\subseteq N, i \neq j \}$ is the set of edges (bilateral communication links)

- A communication graph $(N,L)$ should be interpreted as a way to model restricted cooperation:
  - Players can cooperate with each other if they are connected (directly, or indirectly via a path)
  - Indirect communication between two players requires the cooperation of players on a connecting path.
Communication situations (Myerson (1977))

- A *communication situation* is a triple \((N,v,L)\)
  - \((N,v)\) is a n-person TU-game (represents the economic possibilities of coalitions)
  - \((N,L)\) is a communication network (represents restricted communication possibilities)
- The *graph-restricted game* \((N,v^L)\) is defined as

\[
v^L(T) = \sum_{C \in T \setminus L} v(C)
\]

For each \(S \in 2^N \setminus \{\emptyset\}\).

Recall that \(T \setminus L\) is the set of maximal connected components in the restriction of graph \((N,L)\) to \(T\).
Example
A weighted majority game ({1,2,3},v) with the winning quote fixed to \( \frac{2}{3} \) is considered. The votes of players 1, 2, and 3 are, respectively, 40\%, 20\%, and 40\%. Then, \( v(1,3)=v(1,2,3)=1 \) and \( v(S)=0 \) for the remaining coalitions.

The communication network is

```
1 2 3
```

Then,
\[ v^L(1,2,3)=1, \text{ and } v^L(S)=0 \] for the other coalitions.
Solutions for communication situations

- Myerson (1977) was the first to study solutions for communication situations.
- A solution $\Psi$ is a map defined for each communication situation $(N,v,L)$ with value in $\mathbb{IR}^N$.

**PROPERTY 9 Component Efficiency (CE)**

For each communication situation $(N,v,L)$ and all $C \in N \setminus L$ it holds that

$$\sum_{i \in S} \Psi_i(N,v,L) = v(C).$$

- Property 9 is an “efficiency” condition that is assumed to hold only for those coalitions whose players are able to communicate effectively among them and *are not connected to other players.* (maximal connected components)
Solutions for communication situations

**PROPERTY 10 Fairness (F)** For each communication situation \((N,v,L)\) and all \(\{i,j\} \in L\) it holds that 
\[
\Psi_i(N,v,L) - \Psi_i(N,v,L\{\{i,j\}\}) = \Psi_j(N,v,G) - \Psi_j(N,v,L\{\{i,j\}\}).
\]

- Property 10 says that two players should gain or lose in exactly the same way, when a direct link between them is established (or deleted).
Myerson value

Theorem 4 (Myerson (1977))

There exists a unique solution $\mu(N,v,L)$ which satisfies CE and F on the class of communication situations. Moreover,

$$\mu(N,v,L) = \phi(v^L)$$

where $\phi(v^L)$ is the shapley value of the graph-restricted game $v^L$. 
Example
A weighted majority game \(\{1,2,3\},v\) with the winning quote fixed to 2/3 is considered. The votes of players 1, 2, and 3 are, respectively, 40\%, 20\%, and 40\%. Then, \(v(1,3)=v(1,2,3)=1\) and \(v(S)=0\) for the remaining coalitions.

The communication network is

\[1 \rightarrow 2 \rightarrow 3\]

Then,
\(v^L(1,2,3)=1\), and \(v^L(S)=0\) for the other coalitions.

We have that
\(\phi(v)=(1/2,0,1/2)\) and \(\mu(N,v,L)=\phi(v^L)=(1/3,1/3,1/3)\).
**Example** \((N,v,L)\) communication situation such that \(L\) is the following network and \(v=\mathbb{u}_{\{2,4\}}\)

\[
\begin{array}{c}
1 \\
\hline
3 \\
\hline
4 \\
\hline
2
\end{array}
\]

Note that, for instance, \(v^L(2,4)=v(2)+v(4)=0\).

Easy to note that that \(v^L=\mathbb{u}_{\{1,2,4\}}+\mathbb{u}_{\{2,3,4\}}-\mathbb{u}_N\)

Therefore,

\[
\mu(N,v,L)=\phi(v^L)=(1/3,2/3,1/3,2/3)-(1/4,1/4,1/4,1/4)
\]

\[
=(1/12,5/12,1/12,5/12)
\]
Application to social networks

An application of the Shapley value, which uses both the classical one and the one by Myerson (1977), has been proposed by Gómez et al. (2003), to provide a definition of *centrality* in social networks.

The proposal is to look at the difference between:

- \( \mu(N,v,L) \): the Myerson value, that takes into account the communication structure;
- \( \phi(v) \): the Shapley value, that disregards completely the information provided by the graph \( L \).
Games and Centrality

- The centrality of a node refers to the variation in power due to the social situation (represented by the graph),
- the power is measured using game theory
- More precisely, it is the Shapley value of a game that is used as a power index.
- Gómez et al. (2003) describe general properties of their centrality measure, and in particular, how the abstract structure of the graph influences it.
Example
A weighted majority game \((\{1,2,3\},v)\) with the winning quote fixed to \(2/3\) is considered. The votes of players 1, 2, and 3 are, respectively, 40\%, 20\%, and 40\%. Then, \(v(1,3)=v(1,2,3)=1\) and \(v(S)=0\) for the remaining coalitions.

The communication network is

\[
\begin{align*}
1 & \quad 2 \quad 3
\end{align*}
\]

We have seen that
\[
\phi(v)=(1/2,0,1/2) \quad \text{and} \quad \mu(N,v,L)=\phi(v^L)=(1/3,1/3,1/3).
\]
So, the centrality value is \(1/3\) for player 2 and \(-1/6\) for 1 and 3.