WPS4072

COOPERATIVE GAME THEORY AND ITS APPLICATION TO NATURAL, ENVIRONMENTAL AND WATER RESOURCE ISSUES:

1. Basic Theory

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Abstract

Game theory provides useful insights into the way parties that share a scarce resource may plan their utilization of the resource under different situations. This review provides a brief and self-contained introduction to the theory of cooperative games. It can be used to get acquainted with the basics of cooperative games. Its goal is also to provide a basic introduction to this theory, in connection with a couple of surveys that analyze its use in the context of environmental problems and models. The main models (bargaining games, transfer utility and non transfer utility games) and issues and solutions are considered: bargaining solutions, single-value solutions like the Shapley value and the nucleolus, and multi-value solutions such as the core. The cooperative game theory (CGT) models that are reviewed in this paper favor solutions that include all possible players and ignore the strategic stages leading to coalition building. They focus on the possible results of the cooperation by answering questions such as: Which coalitions can be formed? And how can the coalitional gains be divided in order to secure a sustainable agreement? An important aspect associated with the solution concepts of CGT is the *equitable and fair* sharing of the cooperation gains.

World Bank Policy Research Working Paper 4072, November 2006

This Working paper is a product of the first phase of the study "Cooperative Arrangements for Allocation of Water among Production and Environmental Uses under Stochastic Supply Conditions", funded by the Italian Trust Fund and the Development Economics Research Group of the World Bank. Members of the study team include: Ariel Dinar (Task Manager), Fioravante Patrone, Irene Parrachino, Stefano Zara of the University of Genoa, Carlo Carraro, Carmen Marchiori, and Alessandra Sgobbi of the University of Venice. Comments by Rashid Sumaila, University of British Columbia, Canada and Kim Hang Pham Do, Massey University, New Zealand, and participants of the 6th meeting on Game Theory and Practice Dedicated to Development, Natural Resources and the Environment, July 10-12, Zaragoza, Spain, are much appreciated. Two additional Working Papers are also devoted to some special topics, which are chosen having in mind their use in applications to environmental issues: 2. Application to Natural and Environmental Resources (PRWP 4073), and 3. Application to Water Resources (PRWP 4074).

CONTENTS

A SEMI-TECHNICAL BACKGROUND

Resource scarcity is a growing concern to individuals as well as to governments. Competition over smaller amounts of water and other environmental amenities of deteriorating quality is increasingly becoming common in many parts of the world. As more have to share less, the question of strategic behavior becomes imminent. In such a situation, Game Theory (GT) may provide useful insights into the way parties that share a scarce resource should plan their utilization of the resource under different situations.

The purpose of this working paper is twofold. First it provides the basic building blocks of Cooperative Game Theory (CGT), presented in simple and applicable terms. Second, it reviews the literature dealing with application of cooperative game theory in environmental and water resources and explores the possibilities of expanding its use. The paper does not provide a general introduction to GT, or even to CGT: it is just a sketchy introduction, focusing on the tools that have been used in applications to water and environmental problems. The goal is to offer a self-contained introduction, instrumental to the reading and understanding of companion surveys, for people who do not have a general GT background.

What is Game Theory?

Game Theory (hereafter GT) is the study of mathematical modeling of strategic behavior of decision makers (players), in situations where one player's decisions may affect the other players. The basic assumption of Game Theory is that decision makers are rational players, that they are intelligent, so, while pursuing well-defined objectives, players take into account other decision-makers' rationality and, accordingly, build expectations on their behavior.

GT consists of a modeling part and a solution part. Mathematical models of conflicts and of cooperation provide strategic behavioral patterns, and the resulting payoffs to the players are determined according to certain *solution concepts*.

There are two main branches of GT.[1] The first is *non-cooperative Game Theory* (hereafter NCGT) and the second is *cooperative Game Theory* (hereafter CGT). The main distinction between the two is that NCGT models situations where players see only their own strategic objectives and thus binding agreements among the players are not possible, while CGT actually is based mainly on agreements to allocate cooperative gains (*solution concepts*).

Therefore, while NCGT models describe and take into account the strategic interaction among the players, CGT ignores the strategic stages leading to coalition building and focuses on the possible results of the cooperation. CGT attempts answering questions such as which coalitions can be formed? And how can the coalitional gains be divided in order to secure a sustainable agreement? In particular, CGT favors solutions that include all possible players (*Grand Coalition*), and thus most CGT solution concepts refer to the Grand Coalition.

An important aspect associated with the solution concepts of CGT is the *equitable and fair* sharing of the cooperation gains. Young (1994) notices that equity is something dealt with in everyday life. One can refer to equity in a comprehensive framework, that is, social justice: a proper distribution of resources, welfare, rights, duties, opportunities, or in the narrow framework, for example, how to solve everyday distributive problems. This second case is the one more frequently addressed by GT, which provides the tools to examine equity in a

[1] Additional typologies hold as well, e.g., static and dynamic games, one-shot games and repeated games.

rigorous way, and the problem turns out to be a choice between rules under an axiomatic perspective. But, as Young underlines, the axiomatic approach has two weaknesses: first, the axioms, reasonable by themselves, may lead to "impossibility theorems"; second, the axiomatic method may result in a solution that is too far from the practical problem dealt with: the perceived equity always depends on the particulars of the case. Furthermore, the *empirical rules* of equity, that one can see applied in real situations, are usually more complex than a single normative principle, and often represent a balance or compromise between competing principles.

The term fairness in the literature is sometimes used as a synonym for equity, but some authors often mean something different: their idea of fairness coincides with the acceptability and stability of the cost-benefit apportionment among the players.

As was indicated above, the following background is not intended to provide a comprehensive technical basis in GT. Readers who wish to widen their familiarity with the field of GT could refer to Owen (1995), Myerson (1991), Osborne and Rubinstein (1994), Driessen (1988), Peters (1997), and Aumann and Hart (1992, 1994, 2002).

THE COOPERATIVE BARGAINING PROBLEM

A GT approach to *bargaining* was first introduced in Nash (1950), who developed a cooperative and static game model. In a *two-person bargaining problem,* two players have access to a set of alternatives, which is called the *feasible set*. It is assumed that each player has preferences over the alternatives, and that the preferences are represented by a couple of functions u_1 and u_2. In the original approach by Nash, as in many later developments, it has been assumed that these utility functions are von Neumann-Morgenstern utilities[2].

The utility functions, u_1 and u_2, define a subset S of R^2, which is the image of the set of feasible alternatives. Each point of S represents a solution for the bargaining problem that is an agreement between the players. Note that the solution is defined in the 'image' space, so that in principle it could correspond to more different 'equivalent' agreements.

Within the feasible set there is also a point, called *threat point* or *disagreement point*, which is where "the game ends" should no agreement be reached. We will call S the feasible set and d the disagreement point.

While defining his axiomatic solution, Nash wanted it to be a rule that associates a point of S with each problem (S, d). To develop it, we will introduce the formal model (and an example), then we will report the Nash axiomatic solution and finally we will present some of the other solution concepts proposed after Nash.

Definition: *A two-person bargaining problem* is a pair (S, d) such that

S is convex, closed (it contains its boundary) and a comprehensive[3] subset of R^2;

$d \in S$, and there exists at least one $x \in S$ such that $x > d$ [4];

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[2] von Neumann-Morgenstern utility: An axiomatic extension of the ordinal concept of utility to uncertain payoffs. An agent characterized by a von Neumann-Morgenstern utility function ranks uncertain payoffs according to (higher) expected value of the utility of the individual outcomes that may occur.

[3] $\forall x \in S \subseteq R^2, \quad \forall y \in R^2, y \le x \Rightarrow y \in S$

$S_d := \{x \in S \mid x \geq d\}$ is a compact set.

We'll call $\mathcal{B}_2$ the set of two-person bargaining problems.

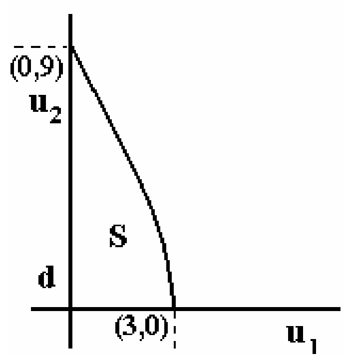
EXAMPLE: dividing wine (no money admitted).

A father says to his two sons: "Here is a container of 9 liters of wine; if you come to an agreement in order to divide the wine, then it is yours, otherwise I'll take the container back."

The possible shares are $(t, 9-t)$, $0 \leq t \leq 9$. Assume that an amount *x* has utility $u_1(x)$ for the first son, and an amount *y* has utility $u_2(y)$ for the second one, with:

$$u_1(x) = \sqrt{x}$$
$$u_2(y) = y$$

A generic splitting $(t, 9-t)$ provides the couple of utility values $(\sqrt{t}, 9-t)$. This situation leads to a bargaining problem (*S*, *d*), where *d* = (0, 0) (no wine) and $S = \text{compr}\left\{\left(\sqrt{t}, 9-t\right) : 0 \leq t \leq 9\right\}$.



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Definition: A *solution* for two-person bargaining problems is a map ϕ: $\mathcal{B}_2 \to \mathrm{R}^2$ satisfying:

$\phi(S,d) \in S$, for all $(S,d) \in \mathcal{B}_2$;

$\phi(S,d) \geq d$, for all $(S,d) \in \mathcal{B}_2$.

A solution associates with each bargaining element $(S,d) \in \mathcal{B}_2$ a unique point of S interpreted as a prediction, or recommendation, for that bargaining problem.

In the axiomatic characterization of his solution concept, Nash required some properties and proved that they characterize only one solution, now called the Nash solution of bargaining problems.

[4] $x = (x_1, x_2)$; $d = (d_1, d_2)$; $x, d \in \mathrm{R}^2$. Then, $x > d$ if $x_1 > d_1$ and $x_2 > d_2$

The main bargaining solutions

The Nash solution

As mentioned before, the Nash bargaining solution is based on several axioms:

1. Individual Rationality. Each player receives at least what he would receive in the threat point, that is, for all $(S,d) \in \mathcal{B}_2$ we have $\phi(S,d) \geq d$;

2. Pareto optimality. This is a standard efficiency condition and means that all gains from cooperation should be shared. Formally, it is expressed as follow:

ϕ satisfies the Pareto optimality if, for each $(S,d) \in \mathcal{B}_2$ and for each $x \in S$, $x \geq \phi(S,d)$; then $x = \phi(S,d)$;

3. Independence from Irrelevant Alternatives. If an alternative is judged to be the solution to a problem, then it should still be considered the most suitable for any sub-problem containing it. Formally, it is expressed as follows:

ϕ satisfies this property if $\phi(S,d) \in T$ implies $\phi(T,d) = \phi(S,d)$, for each $(S,d) \in \mathcal{B}_2$ and for each $T \subset S$ such that $(T,d) \in \mathcal{B}_2$;

4. Covariance under positive affine transformations. The solution is required to be independent of any particular members in the families of von Neumann and Morgenstern's utility functions (von Neumann and Morgenstern, 1944) representing the agents' preferences chosen to describe the problem. Formally, it is expressed as follows:

A positive affine transformation is a map A: $\mathrm{R}^2 \to \mathrm{R}^2$ such that, for every $i \in \{1,2\}$, $A_i(x) = a_i x_i + b_i$,

where $a_1, a_2, b_1, b_2 \in \mathrm{R}$, a_1 and a_2 being positive ($A_i(x)$ denotes the *i*-th component of $A(x)$).

ϕ satisfies covariance with positive affine transformation if, for each $(S,d) \in \mathcal{B}_2$ and for each positive affine transformation A, $\phi(A(S), A(d)) = A(\phi(S,d))$;

5. Symmetry. If the bargaining problem is invariant with interchanging the agents, the solution should give the same to both. Formally, it is expressed as follows:

ϕ satisfies symmetry if, for each $(S,d) \in \mathcal{B}_2$ such that S is symmetric (i.e. $(x_1, x_2) \in S$ then $(x_2, x_1) \in S$) and $d_1 = d_2$, it holds that $\phi_1(S,d) = \phi_2(S,d)$.

Nash proved that there is a unique solution, Ω, satisfying all these axioms, that is expressed in the following formula:

$$\Omega = \max_{S_d} (x_1 - d_1)(x_2 - d_2).$$

No axiom is universally applicable; for a critique to the set of these 5 axioms see Thomson (1994).

The Nash solution has been generalized by Harsanyi (1959) for *n* players with *n* > 2. The Nash-Harsanyi solution, *Z*, to a n-person bargaining game is given below.

$$Z = \prod_{i=1}^{n} (x_i - d_i).$$

The Kalai-Smorodinski solution

Kalai and Smorodinski (1975) thought that the Nash solution does not take sufficiently into account the aspirations of the players: they believed that if the preferences and the utility values of the players change, the compromise point between the agents should not vary. They proposed a solution taking into account the aspiration levels of the players. The property introduced is the following:

<u>Individual monotonicity</u>. It requires that an expansion of the feasible set "in a direction favorable to a particular agent", always benefits the agent. Formally it is expressed as follows:

Let $u_i(S,d)$ be $\max\{x_i \mid x \in S_d\}$ (for $n=2$): if $T \supseteq S$, and $u_j(T) = u_j(S)$, then $\phi_i(T,d) \geq \phi_i(S,d)$, for $i \neq j$.

By simply replacing independence of irrelevant alternatives by individual monotonicity in the list of axioms proposed by Nash, Kalai and Smorodinski obtained the characterization of their proposed solution.

The Kalai-Smorodinski solution (a unique solution satisfying the above axioms) is:

$$\text{For every } (S,d) \in \mathcal{B}_2,\ \phi(S,d) = d + \overline{t}\left(u(S,d) - d\right)$$

$$\text{where } \overline{t} = \max\left\{t \in \mathrm{R} \mid d + t\left(u(S,d) - d\right) \in S_d\right\}.$$

Although there are several solutions to bargaining problems; the two solutions presented here are the most important ones. Other solution concepts in the game theoretical literature can be found in Peters (1992). For example, Armstrong (1994) applies the Salukvadze solution (Yu and Leitman, 1974) to a fishery management problem.

N-PERSON COOPERATIVE GAMES

Definition: An *n-person cooperative game in the characteristic function form* is an ordered pair $G = (N, v)$ where $N = \{1, 2, ..., n\}$ is a finite set with *n* elements.

$N = \{1, 2, ...n\}$ is the set of players. A subset *S* of the player set *N* (notation: $S \subseteq N$) is called *coalition*, and the collection of 2^n coalitions of *N* (i.e. $P(N)$) is denoted by 2^n, including *N* itself, the empty set and all the one-element subsets.

$v : P(N) \to \mathrm{R}$ is real-valued function defined over all the subsets of N and such that $v(\varnothing) = 0$, where $\varnothing$ is the empty set.

N is the *grand coalition* and it consists of the set of all the players. The number of the players of a coalition *S* is denoted by $|S|$ or, sometimes, *s*. v is the *characteristic function*, or coalitional function, and assigns a "worth" $v(S)$ to each coalition *S*.[5]

[5] It is necessary to note that the characteristic function is defined considering only the players within coalition *S*, without considering those players which are outside of the coalition and that, specifically in environmental issues, could affect $v(S)$. For a better explanation, see Section 6.

In many cases the elements of the players' set *N* represent real people, e.g., landowners and peasants, traders, creditors or voters, but the player set can also consist of more abstract membership, such as sectors, as in the well-known Tennessee Valley Authority case (Straffin and Heaney), or (in the "airport game") of landing by planes, or agricultural associations and city water services.

It is often assumed that the coalitional/characteristic function is expressed in units of an infinitely divisible commodity which "stores" utility, and which can be transferred without loss to the players. If a subset of players (coalition) can obtain a total utility v, this utility can be divided among the members of the coalition in any possible way. It is also assumed that there exist some transferable commodity that enables the transfers among players in order to reallocate the benefits gained through the coalition, and that there is a common scale to compare the players' utilities.

The theory developed based on these assumptions is called the *side-payments* theory. We will call games satisfying the above presumptions transferable utility games, or *TU-games*. We will often refer to (N,v) simply as v, and will denote by $G(N)$ the class of TU-games with set of players *N*. The coalitional function is often called a *game* (with transferable utility) *in the characteristic form*.

Definition: In a TU-game $v \in G(N)$ *v* is *superadditive* if, for all $S,T \subseteq N$, with $S \cap T = \varnothing$,

$$v(S \cup T) \geq v(S) + v(T).$$

Definition: Let $v \in G(N)$. *v* is said to be *constant-sum* if, for all $S \subseteq N$,

$$v(S) + v(N \setminus S) = v(N).$$

If a game is superadditive, then players have real incentives for cooperation, in the sense that the union of any two disjoint groups of players shall never diminish the total benefits; the merger of the disjoint coalitions can only improve their prospects. The superadditivity of the game is guaranteed in case $v(S)$ is defined as the total amount of utility that coalition *C* can guarantee for itself, irrespective of the strategies adopted by the remaining players (see the definition of the α - characteristic function given in the last section). It is not always sensible to assume superadditivity, and this assumption may be questioned in some environmental applications, due to the presence of externalities (again, see the last section for some comments on this issue). Assuming superadditivity means that the grand coalition is efficient, thus the problem may turn to be the sharing of the overall utility $v(N)$ among all the players.

In this paper we focus on the class of superadditive games, explicitly specifying if a particular game doesn't fulfill the above condition. A stronger notion than superadditivity is convexity.

Definition: Let $v \in G(N)$. *v* is *convex* if for all coalitions S and T,

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

An equivalent definition is that for any player i, and any coalitions $S \supset T$ not containing i, $v(S \cup i) - v(S) \geq v(T \cup i) - v(T)$. The larger the coalition, the greater the marginal contribution[6] of new members.

[6] For the concept of Marginal Contribution see the section of the Shapley value

With the characteristic function at hand and supposing that the players agree to work together on a certain objective, they will have to divide the total payoff $v(N)$ of the grand coalition. "An allocation problem arises whenever a bundle of resources, rights, burdens, or costs is temporarily held in common by a group of individuals and must be allotted to them individually. An allocation or distribution is an assignment of the objects to specific individuals." (Young, 1994: p.7)

An allocation, usually, requires two different types of decisions: (a) the choice of the total amount of the good to be distributed and (b) the formula or principle of the allocation of that amount. The focus in this review is on the second choice. "An *allocation rule* is a method, process, or formula, which allocates any given *supply* of goods among any potential *group* of claimants according to the salient characteristics of these claimants." (Young, 1994)

A distribution of the amount of $v(N)$ among the players will be represented by a real value function x on the player set N satisfying the efficiency principle:

$$\sum_{i \in N} x(i) = v(N).$$

Here $x(i)$, also denoted x_i, represents the payoff to player i, according to the involved payoff function x.

We usually identify the function $x \in \mathrm{R}^N$ on N with the corresponding n-tuple $x = (x_1, \ldots, x_n) \in \mathrm{R}^N$ of real numbers, called *allocation*. The vectors $x \in \mathrm{R}^n$ satisfying the efficiency principle are called *efficient payoff vectors* or *pre-imputations*.

Most of the proposed solution concepts meet also the individual rationality principle, requiring that the payoff to any player i by the payoff vector x be at least the amount that player i can attain by himself.

This determines the set of imputations:

Definition: Let $v \in G(N)$. An *imputation* of v is a vector $x \in \mathrm{R}^n$ such that:

$\sum_{i \in N} x_i = v(N)$, (efficiency)

$x_i \geq v_i$, for all $i \in N$. (individual rationality)

$$\mathrm{I}(\mathrm{v}) := \left\{ \mathrm{x} \in \mathrm{R}^{\mathrm{N}} \middle| \sum_{\mathrm{i} \in \mathrm{N}} \mathrm{x}_{\mathrm{i}} = \mathrm{v}(\mathrm{N}), \mathrm{x}_{\mathrm{i}} \geq \mathrm{v}(\mathrm{i}) \forall \mathrm{i} \in \mathrm{N} \right\}.$$

$I(v)$ denotes the set of imputations of v. It is the set of allocations of $v(N)$ among the players satisfying efficiency and individual rationality (that is, as we already mentioned, one player joining a coalition should gain at least what he/she would get playing alone).

Applying superadditivity repeatedly (i.e.: adding the players one by one), it is clear that

$$v(N) \geq \sum_{i \in N} v(\{i\}).$$

If $v(N) = \sum_{i \in N} v(\{i\})$, and $x_i \geq v_i$, $\forall i$ (x is an imputation), then $x_i = v_i$, $\forall i$. There is only one imputation and the game is called *inessential*. In this situation, there isn't any incentive to form coalitions because they don't get more than the sum of the stand-alone payoffs. The other games are called *essential*.

An important approach, developed by von Neumann and Morgenstern (1944) deals with the concepts of *dominance*, *D-Core* and *Stable Sets*. We couldn't find, in our literature review of CGT and environmental and water resources, any explicit applications of this approach, but still, we have to mention it because of its relevance in GT evolution.

In a game we may find many allocations satisfying the requisites above, namely being imputations, but what we might need is a criterion to make a distinction between different possible imputations. *Dominance* is such a criterion.

Definition: Given $v \in G(N)$, a non-empty $S \subseteq N$, and $x, y \in I(v)$, then x *dominates* y *through S* if:

$x_i > y_i$ for all $i \in S$. This states that all of the members of S prefer x to y; and

$\sum_{i \in S} x_i \leq v(S)$. This states that the players of *S* are able, via cooperation by themselves, to obtain the allocation x.

As we will better observe later, CGT tools are used not only to share benefits from cooperation, but also to allocate costs. In this case it is useful to consider a *cost function c* instead of *v*, and this change doesn't introduce substantial differences in the model. In such a cost allocation situation it's also possible to develop a model dealing with the function *v*, which describes the savings of the players that are the result of cooperation.

The main goal of the theory of TU-games is to select, for every TU-game, an allocation, or a set of allocations, which is admissible to the players.

In the next section we will present some of the most important CGT solution concepts, which have been associated with the applications that we found in the literature on CGT and environmental and water resources.

As we indicated earlier, the questions addressed in a CGT model are:

1. Which coalitions form?
2. If the grand coalition forms, how do the players divide $v(N)$ (or allocate costs)?

The solution concepts give many answers to these questions.

One can find two groups of solution concepts: (a) Subset solution concepts: *Core* (Gillies, 1953; Shapley, 1971), *D-Core*, *Stable sets* (von Neumann and Morgenstern, 1944), *bargaining set* (Aumann and Maschler, 1964), *Kernel*, *Least Core*, and (b) One-point (unique) solution concepts: *Nucleolus* (Schmeidler, 1969), *Shapley value* (Shapley, 1953); τ -*value* (Tijs, 1981).

Apart from those quoted above, in the GT literature other solutions can be found, or variants. In this paper, we'll confine ourselves to an overview of the CGT solution concepts that have been applied in the literature on environmental and water resources issues.

Subset solution concepts

The solution concepts in this group refer to a 'range' of values that fulfill certain conditions as will be seen below. The flexibility subset solution concepts provide is very convenient in practice as they allow the decision maker consideration of various policy interventions and their evaluation.

The Core

This concept was introduced by Gillies (1953). An allocation satisfying the two properties (efficiency and individual rationality) stated before is an imputation. From a "practical" point of view, an imputation is a distribution of gains that satisfies only the individual rationality. A similar condition for coalitions should also be taken into account.

So, we can introduce another condition, called *coalitional rationality* that extends the concept of individual rationality to each of the coalitions.

Definition: Let $v \in G(N)$. A *Core element* of a game *v* is an allocation $x \in \mathrm{R}^N$ such that:

$\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N$; (coalitional rationality)

$\sum_{i \in N} x_i = v(N)$. (efficiency)

The *Core* of *v*, that we denote *C*(*v*), is the following set of vectors:

$$C(v) := \left\{ x \in R^N \mid \sum_{i \in N} x_i = v(N),\ and\ \sum_{i \in S} x_i \geq v(S),\ \forall S \subseteq N \right\}$$

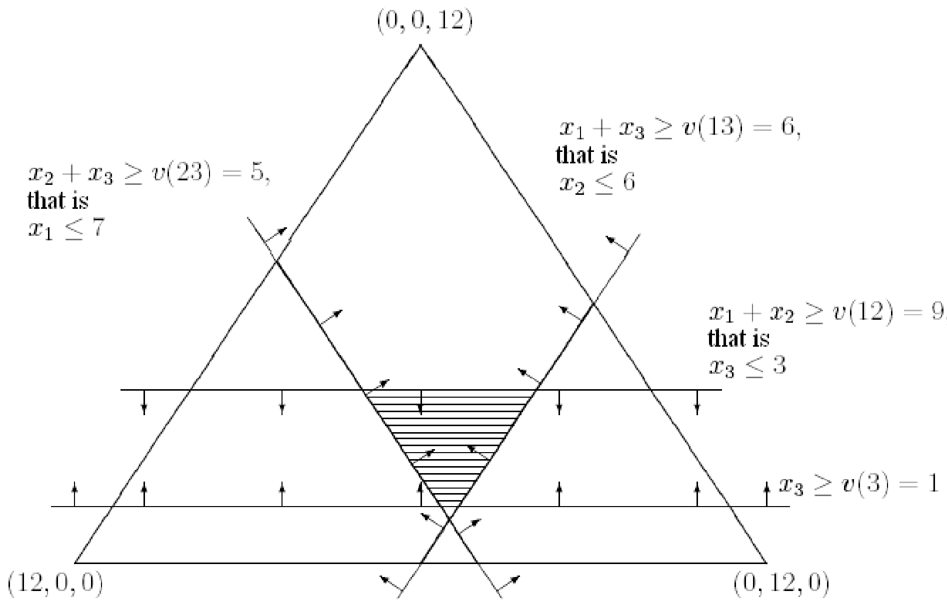
Core allocations provide incentives for cooperation. If the Core exists, however, there are usually an uncountable number of allocations. Which is the most equitable? Thus, two problems may arise: first, the Core may have more than one allocation; second the Core might be empty. The two following examples underline these problems.

EXAMPLE: a game with a nonempty Core.

Consider the game $G = (N, v)$ with $N = \{1, 2, 3\}$ and *v* defined as follows:

$$\begin{aligned} &v(\varnothing) = v(1) = v(2) = 0 \\ &v(3) = 1 \\ &v(1,2) = 9 \\ &v(2,3) = 5 \\ &v(1,3) = 6 \\ &v(1,2,3) = 12 \end{aligned}$$

The simplex in R^3, with vertexes (12, 0, 0), (0, 12, 0), (0, 0,12) represents the part of the *pre-imputation* set $(\sum_{i=1}^{3} x_i = 12)$ which satisfies $x_i \geq 0$. Individual rationality is already satisfied for players 1 and 2, but to get the imputation set we have to consider the condition for player 3 also (see the line $x_3 \geq v(3) = 1$ in the figure below).



(The arrows stand for the direction of the inequalities.)

Source: Authors

The set of imputations consists of all $x \in \mathrm{R}^3$ such that:

$$\begin{cases} x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 1 \\ x_1 + x_2 + x_3 = 12 \end{cases}$$

The Core (lined area) consists of all imputations, which further satisfy:

$$x_1 + x_2 \geq 9$$
$$x_1 + x_3 \geq 5$$
$$x_2 + x_3 \geq 6$$

We know that $x_1 + x_2 + x_3 = 12$, thus we can substitute $x_1 + x_2 = 12 - x_3$ (and similarly for the other two inequalities). We will get:

$$\begin{cases} x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 1 \\ x_3 \leq 9 \\ x_2 \leq 5 \\ x_3 \leq 6 \\ x_1 + x_2 + x_3 = 12 \end{cases}$$

that is exactly what we can see represented in the above figure.

EXAMPLE: a game with an empty Core.

A typical example of a game with an empty Core is the *simple majority game*. A coalition wins if it is composed by more than half of the participants, $N=\{1,2,....,n\}$. Then:

$$v(S)=\begin{cases}1 \text{ if } |S|>\dfrac{n}{2}\\ 0 \text{ if } |S|\leq\dfrac{n}{2}\end{cases}$$

The Core conditions for this game provide no solutions.

For example, if we consider a three-person simple majority game we will have: $N=\{1,2,3\}$ and

$$\begin{aligned}&v(\varnothing)=v(1)=v(2)=v(3)=0\\&v(1,2)=v(1,3)=v(2,3)=1\\&v(1,2,3)=1\end{aligned}$$

The Core will be defined by the following system:

$$\begin{cases}x_1\geq 0\\x_2\geq 0\\x_3\geq 0\\x_1+x_2\geq 1\\x_1+x_3\geq 1\\x_2+x_3\geq 1\\x_1+x_2+x_3=1\end{cases}$$

There is no (x_1,x_2,x_3) satisfying the above conditions. In fact, if we sum the third, fourth and fifth inequality we get:

$$2(x_1+x_2+x_3)\geq 3,$$

that is,

$$(x_1+x_2+x_3)\geq\frac{3}{2},$$

and this is impossible because of the last Core condition (efficiency).

Bondareva (1963) and Shapley (1967) provided necessary and sufficient conditions for a game to have a nonempty Core in terms of balanced conditions.

Theorem: (Bondareva, 1963 and Shapley, 1967) A TU-game has nonempty Core if and only if it is *balanced*[7].

Another well-known relation between convexity of games and the existence of Core elements is as follows:

Theorem: (Shapley, 1971) If a game (N,v) is convex, then $C(v)$ is a nonempty set.

[7] Entering into a detailed analysis of the concept of balancedness, is not useful in this paper.

Note that convexity implies that the game has a nonempty Core, but the reverse is not true.

The bargaining set

The *bargaining set* was introduced by Aumann and Maschler (1964). Here we follow Maschler (1992). It is necessary to start form this concept, in order to better understand two other game theoretic concepts: the *Kernel* and the *Nucleolus*.

The idea is that it is necessary to have a process through which it is possible to allocate the gains from cooperation when the grand coalition has not been formed, but we have a certain *coalition structure*:

Definition: Let (N,v) be a TU-game. A *coalition structure* is a partition $\mathcal{B}=\{\mathcal{B}_1,\ldots,\mathcal{B}_k\}$ of the n players, i.e., $\bigcup_{i=1}^{k}\mathcal{B}_i = N$ and for all $i \neq j$, $\mathcal{B}_i \cap \mathcal{B}_j = \varnothing$.

When a particular coalition structure is formed, it is necessary to define an appropriate payoff configuration, satisfying individual and group rationality.

We will use the notation $x(\mathcal{B}_i)=\sum_h x_h$, with $h\in\mathcal{B}_i$, to indicate the sum of the payoffs of the players in a certain coalition $\mathcal{B}_i$.

Definition: An *imputation* for a coalition structure $\mathcal{B}$ is a payoff vector $x=(x_1,x_2,\ldots,x_n)$ satisfying:

$x(\mathcal{B}_i) = v(\mathcal{B}_i)$, for all $\mathcal{B}_i$ in $\mathcal{B}$ (coalitional rationality);

$x_i \geq v(\{i\})$, for all i in N (individual rationality).

We denote by $X(\mathcal{B})$ the space of all the imputations for the coalition structure $\mathcal{B}_i$.

The players belonging to each coalition try to reach their best outcome advancing proposals about their needs, according with the imputation.

Definition: Let x be an imputation in a game (N,v) for a coalition structure $\mathcal{B}$. Let k and l be two distinct players in a coalition $\mathcal{B}_i$ of $\mathcal{B}$. An *objection* of k against l at x is a pair (S,y), satisfying:

$$S \subset N, \quad k \in S, \quad l \notin S;$$
$$y \in \mathrm{R}^S, \quad y(S)=v(S); \quad y_i > x_i, \quad \text{all } i \in S.$$

This is a way for k to inform l that he could gain more someplace else, and to say to l that he should transfer some of its gains to k.

But l can answer with a counter-objection:

Definition: Let (S,y) be an objection of k against l at x, $x\in X(\mathcal{B})$, $k,l\in\mathcal{B}_i\in\mathcal{B}$. A counter-objection to this objection is a pair (T,z), satisfying:

$T \subset N,\ l \in T,\ k \notin T$;

$z \in \mathrm{R}^T,\ z(T)=v(T)$;

$z_i \geq y_i$, all i in $\mathrm{T}\cap S$;

$z_i \geq x_i$, all i in $T \setminus S$.

In the counter-objection, player l claims that he can assure to himself his gain by forming T . It is important to note that k can object against l only if they belong to the same coalition of the coalition structure. The objection is justified if it has no counter-objection; otherwise, the objection is unjustified.

The bargaining set will be defined as follows:

Definition: Let (N, v) be a cooperative game with side payments. The *bargaining set* $M_1^i(\mathcal{B})$ for a coalition structure $\mathcal{B}$ is:

$M_1^i(\mathcal{B})$:= {$x \in X(\mathcal{B})$: every objection at x can be countered}

= {$x \in X(\mathcal{B})$: there exists no justified objection at x}.

When a particular coalition structure has formed, the aim is to find a particular imputation in order to create some "stability" among the coalitions in the coalition structure.

To compute, at least partially, the bargaining set, the Kernel has been introduced. If the game is superadditive, then the Kernel is non-empty and it is a subset of the bargaining set.

The Kernel

For the *n*-person game *v*, let *S* be a coalition ($S \subseteq N$) and $x = (x_1, ..., x_n)$ a payoff vector (not necessarily an imputation).

We define *excess* of *S* with regards to the imputation *x* as the quantity

$$e(S, x) = v(S) - \sum_{i \in S} x_i \ .$$

corresponding to the difference between the value of the coalition *S* (given by the characteristic function) and the utility received according to *x*.

If we consider two different players ($i \neq j$) the *surplus* of *i* against *j* can be defined as:

$$s_{ij}(x) = \max e(S, x),$$

where the maximum is taken over all coalitions *S* such that $x \in S$ and $j \notin S$.

Thus, s_{ij} represents the maximum gain that player *i* can hope to get without the cooperation of *j*.

Now, if we consider an individually rational payoff configuration *x*, we can say that *i* *outweighs* *j* (notation $i \succ\succ j$) if and only if:

$$s_{ij}(x) > s_{ji}(x) \text{ and } x_j > v(j).$$

If $i \succ\succ j$ there is a certain instability, because *i* can make *j* a demand that he cannot counter.

The Kernel is the set of individually rational payoff configurations for which such instability does not occur, that is $\forall S \subseteq N$, there are no $i, j \in S$, such that $i \succ\succ j$.

The Least Core

Given an n-person game v, we can consider any coalition $S \subseteq N$ and any imputation $x = (x_1, ..., x_n) \in I(v)$; the *excess* of S with regard to the imputation x is described by the quantity

$$e(S, x) = v(S) - \sum_{i \in S} x_i \ .$$

Let $e_1(x)$ be the largest excess of any coalition relative to x, $e_2(x)$ the second largest excess, $e_3(x)$ the next and so on.

The *Least Core* is the set X_1 of all x that minimize $e_1(x)$.

Unique (One-point) solution concepts

Under this group of solution concepts we find several solution concepts such as the Nucleolus, the Kernel, the Shapley Value, and the τ-value.

The Nucleolus

Starting from the definition of the Least Core, let X_2 be the set of all x in X_1 that minimize $e_2(x)$, X_3 the set of all x in X_2 that minimize $e_3(x)$, …

This process will eventually lead to an X_k consisting of a single imputation x (as Schmeidler proved), called the *Nucleolus*.

We can better understand and formalize this concept defining the 2^n-vector $\theta(x)$, as the vector whose components are the excesses of the 2^n subsets $S \subseteq N$, arranged in decreasing order, i.e.,

$$\theta_k(x) = \left(e(S_k, x)\right)_{S \subset N}$$

where $S_1, S_2, ..., S_{2^n}$ are the subsets of N arranged by $e(S_k, x) \geq e(S_{k+1}, x)$.

EXAMPLE: For the three person game v, where $v(S) = 1$, if S has two or three players and $v(S) = 0$ otherwise, the payoff vectors x = (0.3, 0.5, 0.2) and y = (0.1, 0.5, 0.4) give the following excesses:

S	$e(S, x)$	$e(S, y)$
$\varnothing$	0	0
1	- 0.3	- 0.1
2	- 0.5	- 0.5
3	- 0.2	- 0.4
1, 2	0.2	0.4
1, 3	0.5	0.5
2, 3	0.3	0.1
N	0	0

We get $\theta(x) = (0.5, 0.3, 0.2, 0, 0, -0.2, -0.3, -0.5)$

and $\theta(y) = (0.5, 0.4, 0.1, 0, 0, -0.1, -0.4, -0.5)$.

We can order the several vectors $\theta(x)$ by the lexicographic order. The lexicographic order can be applied comparing the first components of two vectors: if they are equal, compare the seconds and so on, until we find two different components. The (lexicographically) bigger vector will be the one containing the higher of these two different components. Formally, given two vectors $\alpha = (\alpha_1, ..., \alpha_q)$ and $\beta = (\beta_1, ..., \beta_q)$, we say that α is lexicographically smaller than β if there is some integer k, $1 \le k \le q$, such that:

$\alpha_l = \beta_l$ for $1 \le l \le k$,

$\alpha_k < \beta_k$

and we can write $\alpha <_L \beta$. (In the above example $\theta(x) <_L \theta(y)$.)

The *Nucleolus* is the imputation that minimizes the function $\theta(x)$ (in the lexicographic order). If the Core is non-empty, then it can be proved that the Nucleolus lies in the Core.

The Shapley value

Several solution concepts are introduced in an *axiomatic* way (as we have seen in Nash and Kalai-Smorodinski): it means that the solution is built as a consequence of some properties that must be satisfied. Usually these properties are applied to a class of games. For this reason, we consider $G(N)$, the class of superadditive games with a given set N of players. We shall assume the N is a finite set containing n elements.

Shapley proposed to consider a mapping $\phi : G(N) \to \mathrm{R}^n$, satisfying the following properties:

Efficiency: ϕ satisfies efficiency if, for all $v \in G(N)$, $\sum_{i \in N} \phi_i(v) = v(N)$. This property means that all the available amount must be allocated by the solution;

The Dummy player property: ϕ satisfies this property if, for every $v \in G(N)$ and every dummy player $i \in N$, $\phi_i(v) = v(i)$.

A player is a *dummy player* if $v(S \cup \{i\}) = v(S) + v(i)$ for all coalition S such that $i \notin S$, that is its contribution to S is only $v(i)$. (A special case of dummy player is a *null player*: a null player does not generate any benefits and should receive nothing);

Anonymity: ϕ satisfies anonymity if, for every $v \in G(N)$ and every $i, j \in N$ that are interchangeable[8] players in v, it holds that $\phi_i(v) = \phi_j(v)$. This property means that the value must treat equally strong players in an equal way;

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[8] If two players equivalently contribute in the game, we say that they are interchangeable. Formally, $i, j \subset N$ are said to be interchangeable players in the game v if, for every coalition $S \subset N \setminus \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{i\})$

Additivity: ϕ satisfies additivity if, for every $v, w \in G(N)$, it holds that $\phi(v+w) = \phi(v) + \phi(w)$. It is difficult to justify this axiom with regards to fairness, but it is an obviously useful property for the solution.

Shapley proved that there exists a unique value $\phi : G(N) \to \mathrm{R}^n$ that satisfies those four properties:

$$\phi_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \left(v\left(S \cup \{i\}\right) - v(S)\right),$$

for all $v \in G(N)$ and all $i \in N$, when *s* and *n* denote the cardinality of *S* and *N*, respectively.

We can interpret the Shapley value as follows: given a game $v \in G(N)$, we consider any permutation π of the set *N* and any player $i \in N$. If $P(i, \pi)$ is the set of players that precede *i* in the permutation π, $M(i, \pi) = v\left(P(i, \pi) \cup \{i\}\right) - v\left(P(i, \pi)\right)$ is the *marginal contribution* of *i* to the coalition $P(i, \pi)$.

The Shapley value $\phi : G(N) \to \mathrm{R}^n$ will be:

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi} M(i, \pi).$$

In order to better understand this concept, consider a situation with *n* players agreeing to meet in a certain room; imagine the *n* players entering one at a time into that room in a random order (specified by the permutation π, that is, all possible arrival orderings are equally probable) and that each player, as soon as he enters and reaches the coalition *S* created by the players arrived before him, receives a reward equal to $v\left(S \cup \{i\}\right) - v(S)$, that is his marginal contribution.

The Shapley value is the mean marginal contribution, averaged on all of the *n!* permutations π.

EXAMPLE: Consider the three-person game:

$N = \{1, 2, 3\}$

$V(1) = V(2) = V(3) = 0$

$V(1,\ 2) = 4;\ V(1,3) = 7;\ V(2,3) = 15$

$V(1, 2, 3) = 20$

The marginal contributions are given in the following table:

π	$M(1, \pi)$	$M(2, \pi)$	$M(3, \pi)$
(1, 2, 3)	0	4	16
(1, 3, 2)	0	13	7
(2, 1, 3)	4	0	16
(2, 3, 1)	5	0	15
(3, 1, 2)	7	13	0
(3, 2, 1)	5	15	0

The average of the sum of the marginal contributions will be $\frac{1}{6}(21,45,54)$, that is, by definition, the Shapley value for this game.

Theorem: Let (N,v) be a TU-game. Then the following statements are equivalent:

The game is convex (that is, for any $S,T \subseteq N$, $v(S)+v(T) \le v(S \cap T)+v(S \cup T)$);

For each permutation π, the marginal contribution vector is in the Core.

The Shapley value, being the mean marginal contribution averaged on the permutation π in all of the *n!* possible cases, is in the Core of a convex game.

<u>The τ - value</u>

The τ - *value* is a feasible compromise between two vectors: *M*(*v*), that is the *marginal vector* (*utopia vector*) and *m*(*v*), that is the *minimum right vector* (see below for further explanation).

For player *i* we define:

$$M_i(v) := v(N) - v(N \setminus \{i\})$$

(Marginal contribution of player *i* to the grand coalition. It is the best payoff player *i* can hope to obtain: if he wants more, then it is preferable for the other players in *N* to throw him away.)

$$m_i := \max_{S:i\in S}\left(v(S) - \sum_{j\in S\setminus\{i\}} M_j(v)\right)$$

(Minimum right for player *i*, called the remainder for *i* in the coalition *S*, if all the other players in *S* obtain their respective utopia payoffs.)

One can prove that $\forall x \in C(v),\ m_i(v) \le x_i \le M_i(v),$ for each $i \in N$, that is *m* and *M* are upper and lower bounds, respectively, for the Core of the game.

As we saw in the section about the Core, games with a nonempty Core are balanced.

A game is called *Quasi Balanced* (QB) if:

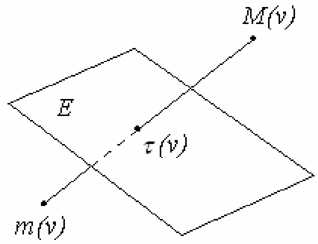
$m_i(v) \le M_i(v) \quad \forall i \in N$

$\sum_{i\in N} m_i(v) \le v(N)$

$\sum_{i\in N} M_i(v) \ge v(N)$

One can prove that each balanced game is quasi balanced.

For a QB game, $\tau(v)$ is the unique efficient payoff vector on the line segment $[m(v), M(v)]$, lying in the *hyperplane* *E* of efficient payoff vectors $E = \left\{x \in \mathrm{R}^N \middle| \sum_{i=1}^n x_i = v(N)\right\}$.



Source: Authors

Modifications to the above solution concepts

Many modifications of the above solution concepts have been applied in the environmental CGT literature and some of them even were developed from a case study.

The *weak Least Core* (Hashimoto et al., 1982) (and, equivalently, the *weak Nucleolus*) is obtained by finding the least ε for which there is a solution *x* to the system:

$$\sum_{S} x_i \geq v(S) - \varepsilon|S| \quad \forall S \subseteq N$$
$$\sum_{N} x_i = v(N)$$

This might be interpreted as imposing a minimum uniform tax on any individual player that joins a coalition.

The *proportional Least Core* (Hashimoto et al., 1982) (and the *proportional Nucleolus*) can model a similar situation, where we have to impose a minimum tax (or subsidy) on each coalition proportionally to the respective costs associated with this coalition (in a cost game). Thus we can postulate a tax rate *t* and the system to be solved will be:

$$\sum_{S} x_i \geq (1-t)v(S) \quad \forall S \subseteq N$$
$$\sum_{N} x_i = v(N)$$

Another modified Nucleolus is the *Normalized Nucleolus* (Davos and Lejano, 1995): the excess is substituted by the quantity $\frac{e(S,x)}{x(S)}$, which stands, for example, for the economical rate of return of joining a coalition. Rogers (1994) refers to the weak Nucleolus as the *per-capita Nucleolus*.

Also for the Shapley value many variants have been proposed. Of all the properties that characterize the Shapley value, symmetry seemed to be the most "innocuous". Yet from a modeling point of view, this assumption is perhaps the trickiest because it calls for a judgment about what should be treated equally.

Furthermore in the Shapley case all $S \subseteq N$ are assumed to be possible, and all the orderings of users joining a coalition are equally likely; this assumption also might be unrealistic.

These are the reasons why the *Weighted Shapley value* (*Generalized Shapley value*) has been introduced, taking into account the probability of formation for the coalitions [Loehman and

Whinston (1976) , Shapley (1981) , Kalai and Samet (1987), Young (1994). For a review of the literature on weighted Shapley values see Kalai and Samet (1988).

NON TRANSFERABLE UTILITY (NTU)-GAMES

An NTU-game is a pair (N,V) where $N=\{1,...,n\}$ and where V is a map assigning to each $S \in 2^N \setminus \{\varnothing\}$ a subset $V(S)$ of R^S such that the following three properties hold for each S (Tijs, 2003):

$V(S)$ is a non-empty closed proper subset of R^S;

$V(S)$ is comprehensive, i.e. if $x \in V(S)$ and $y \in \mathrm{R}^S$ such that $y \le x$, then $y \in V(S)$.

$\{x \in \mathrm{R}^n \mid x_i \ge v(i), x \in V(N)\}$ is bounded, where $v(i)$ is the maximum of $V(\{i\})$.

Note that R^S is defined as follows:

$\mathrm{R}^S = \{ f : S \to \mathrm{R}\}$.

This notation is useful in order to distinguish, for example, between $\mathrm{R}^{\{1,2\}}$ and $\mathrm{R}^{\{1,3\}}$. It is possible to turn R^S into R^n setting the coordinates not belonging to *S* equal to zero.

The elements of N are players, who can cooperate. If coalition S forms, then each of the payoff vectors $x \in V(S)$ is attainable, giving reward (utility) x_i to player $i \in S$.

Each *n*-person TU-game (N,v) gives rise to an *n*-person NTU-game (N,V), where

$$V(S) = \left\{ x \in \mathrm{R}^S \mid \sum_{i \in S} x_i \le v(S) \right\}.$$

A two-person bargaining game (S,d) can be seen as a 2-person NTU-game $(\{1,2\},V)$ where

$$V(\{1\}) = (-\infty, d_1], V(\{2\}) = (-\infty, d_2]$$
$$V(\{1,2\}) = S .$$

COST GAMES

In this section we will underline the link between cooperative Game Theory and the analysis of the allocation of costs (and gains) within a group of players.

By *cost-sharing* problem, we usually mean a problem arising when a group has to decide how to allocate costs of a joint enterprise. A cost-sharing problem can be modelled in CGT using a *cost-game*.

To introduce cost games we use a very typical situation of multipurpose joint project. Let $N=\{1,2,...,n\}$ be a set of projects, products, or services (or agents). The projects can be provided jointly or separately by some organization. Let $c(i)$ be the cost of providing *i* by itself, and for each subset $S \subseteq N$ let $c(S)$ be the cost of providing the items to *S* jointly. For convenience, we set $c(\varnothing)=0$. (We assume that if the project is not undertaken, then the cost is zero.) The function $c(S)$ from the subsets S of N to the real numbers is called *cost function* on *N*, or, sometimes, *cost-sharing game*.

On the other hand, the *cost-savings function* (also identified with the corresponding *cost-savings game)* is $v(S)=\sum_{i\in S}c(i)-c(S)$ and it represents the gain from carrying the projects in S jointly, rather than separately by *i*. If $v(N)\geq v(S),\forall S\subseteq N$, then the multipurpose project is efficient and the problem is to allocate the cost $c(N)$ in an equitable way among the projects.

In some contexts it is natural to interpret $c(S)$ as the least costly way to carry out the projects in *S*. If the cost function is interpreted in this way, then for any partition of a subset of projects into two disjoint subsets S' and S'', we have:

$$c(S'\cup S'')\leq c(S')+c(S'').$$

This property is known as *subadditivity* (equivalent to superadditivity for *v*).

A second "natural" property of a cost function is that costs increase as the number of projects increase, that is

$$c(S)\leq c(S') \text{ for all } S\subseteq S'.$$

Such a cost function is called *monotonic*.

If *c* is subadditive, then *v* is nonnegative and monotonically increasing in *S*.

In fact, $\forall S,S'$ and $i\notin S$, subadditivity implies that $c(S+i)\leq c(S)+c(i)$, and it follows that $v(S)\leq v(S+i)$. Since $v(\varnothing)=0$, *v* is nonnegative and monotonic.

It also follows that *v*(*N*) is the largest among all *v*(*S*). In this case, from a formal point of view, *N* is the efficient set of projects to undertake.

Usually in the cost games the terms *allocation* or *solution* for the problem (*c*, *N*) stand for the vector $x=(x_1,\ldots,x_n)\in \mathrm{R}^N$ such that $\sum_{i\in N}x_i=c(N)$ where x_i is the amount charged to project *i*, while in general when we require the efficiency condition, we talk about pre-imputations.

A *cost allocation rule* or *cost sharing rule* is a function $\Phi(c)$ that associates a unique solution to every cost-sharing game.

Cost-sharing rules

"A *cost sharing rule* allocates the total cost of a project among the members of a group that participate in that project for every possible specification of the cost function. An allocation is in the *Core* of the cost-sharing game if no participant, or group of participants, pays more than its stand-alone cost." (Young, 1994).

As we can see, two of the CGT concepts we introduced in the previous sections immediately come out: the *imputation set* and the *Core*.

The Core can be interpreted, in such situations, as an incentive for the parties to cooperate. A first requirement for a cost allocation is that no participant or group of participants be charged more than their stand-alone costs.

Formally:

$$\sum_{i\in N} x_i = c(N)$$

$$x(S) \le c(S), \text{ where } x(S) = \sum_{i\in S} x_i \ .$$

How to select an equitable allocation in the Core? And how to find out a cost allocation if the cost-game has an empty Core?

The above reported solution concepts can obviously be useful in this framework also, but some problems still remain and many studies' results are that no cost-sharing method can be universally preferred.

The Shapley value can be interpreted from a cost-sharing point of view in the following way: "Given a cost-sharing game on a fixed set of players, let the players join the cooperative enterprise one at a time in some predetermined order. As each player joins, the number of players to be served increases. The player's *cost contribution* is his net addition to cost when he joins, that is, the incremental cost of adding him to the group of players that has already joined. The *Shapley value* of a player is his average cost contribution over all possible orderings of the players." (Young, 1994). Here the player's *cost contribution* has to be referred to as his *marginal contribution*.

Then, the Shapley value can also be formulated as

$$\phi_i(c) = \sum_{S \subseteq N-i} \frac{|S|!\left(|N-S|-1\right)!}{|N|!}\left[c(S+i)-c(S)\right].$$

Following is an overview on the main cost sharing rules in the literature (the "classical" ones plus some more recent methods). Almost all the methods charge a minimum cost called *separable cost*, that is, in a multipurpose project, a change is imposed due to adding a project to the set of the others:

$$SC_i = c(N) - c(N-i).$$

If the savings game is convex it follows that

$$SC = \sum_{i=1}^{n} SC_i \le c(N).$$

and the remaining allocation problem turns to be the allocation of the *non-separable costs*:

$$NSC = c(N) - SC \ .$$

A possibility is to equally allocate the NSC among the projects (Egalitarian Non separable Cost Method, or ENSC)

$$x_i = SC_i + \frac{NSC}{n}.$$

The Alternate Cost Avoided (ACA) and the Separable Cost Remaining Benefits (SCRB) methods

Another method proposed by a TVA consultant, Martin Glaeser, in 1938, assigns to each project the following charge

$$x_i = SC_i + \frac{c_i - SC_i}{\sum_{j=1}^{n} c(j) - SC_j} NSC .$$

This method attributes to a project its separable cost and a rate of the *NSC* proportional to $c_i - SC_i$. This difference is the *Alternate Cost Avoided*.

A modification of this method is the allocation method, which has gained widest acceptance among water resource engineers and is still in use.

In this modification $c(i)$ is replaced by min $[b(i), c(i)]$ in the following equation

$$x_i = SC_i + \frac{\min\left[b(i), c(i)\right] - SC_i}{\sum_{j=1}^{n} \min\left[b(j), c(j)\right] - SC_j} NSC .$$

This is called *Separable Cost Remaining Benefits method* (James and Lee, 1971) and it incorporates benefits as follows. Let $b(i)$ be the benefit from undertaking project i by itself. Then the *maximum justifiable expenditure* for i is $\min[b(i), c(i)]$.

The ACA method allocates cost savings in proportion to each project's marginal contribution to savings (Straffin and Heaney, 1981).

This solution was proposed independently in the game theory literature as a means of *minimizing players' "propensity to disrupt"* (Gately (1974), Littlechild and Vaidya (1976), Charnes et al. (1979)).

Gately's definition of propensity to disrupt was:

$$d_i = \frac{v(N) - v(N-1)}{x_i} - 1 .$$

This equation represents the ratio of what the members of coalition N - i would loose if player i disrupted the grand coalition, to what player i himself would lose.

Straffin and Heaney showed that Gately's apportionment method, based on minimizing the maximum propensity to disrupt, is exactly the ACA method.

There is no reason to think that the ACA method yields a solution in the Core, and indeed it does generally not. However when there are three projects at most, and the cost function is subadditive, then it is in the Core, provided that the Core is nonempty.

Dickinson and Heaney have proposed a modification to the SCRB (1982): the *Minimum Costs Remaining Savings* (*MCRS*) *method*.

Driessen and Tijs (1986) proposed another cost allocation method, deriving from the τ-value, that is the *Non Separable Cost Gap* (*NSCG*) *method*.

Monotonicity

Up to this point we have implicitly assumed that all cost information is at hand, and the agents need only to reach an agreement on the final allocation.

In practice, however, the parties may need to make an agreement before the actual costs are known, and they commit themselves to a rule for allocating costs rather than to a single cost allocation.

This turns our attention to the type of rule on which the players will agree. For example, if total costs were higher than anticipated, it would be unreasonable for any player's charge to go down. Similarly, if costs were lower, it would be unreasonable for any player's charge to go up.

Formally an allocation rule Φ is *monotonic* (in the aggregate) if for any set of projects *N*, and for any cost functions *c* and *c'* on *N*:

$$c'(N) \geq c(N) \text{ and } c'(S) = c(S) \text{ for all } S \subset N$$

implies

$$\Phi_i(c') \geq \Phi_i(c) \text{ for all } i \in N .$$

We say that the cost allocation method Φ is *coalitionally monotonic* if an increase in the cost of any particular coalition implies, ceteris paribus, no decrease in the allocation to any member of that coalition. That is, for every set of projects *N*, every two cost functions *c* and *c'* on *N*, and every $T \subseteq N$.

In the same way $c'(T) \geq c(T)$ and $c'(S) = c(S)$ for all $S \neq T$

implies

$$\Phi_i(c') \geq \Phi_i(c) \text{ for all } i \in T .$$

There is a theorem (Young, 1985) stating that: for $|\mathrm{N}| \geq 5$ there exists no Core allocation method that is coalitionally monotonic.

α, β AND γ -CHARACTERISTIC FUNCTIONS

As we saw in the previous sections CGT models require defining a real-valued function (the characteristic function) assigning a real number to every coalition of players.

The worth of a coalition is what it can achieve on its own without the cooperation of non-members. If there are no externalities, i.e., if the payoffs of the members of a coalition do not depend on the actions of non-members, the model doesn't have to take into account the actions of the non-members. But if there are externalities, then it is a must.

There are several examples of externalities such as the market externalities and the environmental externalities (Finus, 2003).

If we admit the possibility of binding agreements among a subset *S* of players, there are different ways to get *v*(*S*) and, in general, the characteristic function *v* of the game, depending on the assumptions about the behavior of the players outside the coalition. Thus, in the literature, we find for example the α, β and γ -characteristic functions. (As a consequence, $\alpha -$ and $\beta -$ Core were defined by Aumann (1959).)

In the α - characteristic function it is assumed that when a coalition *S* forms the players outside the coalition $(N \setminus S)$ choose a strategy implying the worst outcome for *S*; it is the so-called *maximin* outcome. Formally:

$$v_\alpha(S) = \max_{x_S \in X_S} \min_{x_{N\setminus S} \in X_{N\setminus S}} \pi_S(x_S, x_{N\setminus S}),$$

where

X_S is the set of possible strategies for coalition S,

$X_{N\setminus S}$ is the set of possible strategies for coalition $N \setminus S$, .

and π_S is the payoff function of coalition S (equal to the sum of payoff functions of players belonging to S).

In the β - characteristic function it is assumed that the players in $N \setminus S$ give more chances to the players in S, and let them reach at least $v_\beta(S)$, that is the minimum among the maximum payoffs that S can obtain once fixed the strategies of the other players.

This is called the *minimax* outcome:

$$v_\beta(S) = \min_{x_{N\setminus S} \in X_{N\setminus S}} \max_{x_S \in X_S} \pi_S(x_S, x_{N\setminus S}).$$

Somehow, the α - characteristic function represents a pessimistic point of view (of the players in S), while the β - characteristic function represents an optimistic perception.

There is also another important interpretation of the behavior of the players outside S, leading to the γ-characteristic function, developed by Tulkens (1979), and Chander and Tulkens (1992 and 1993). In this case it is assumed that players outside S do not play against it choosing the worst strategy for S, but they try to reach the Nash equilibrium of the game, with S considered as a single player. It is important to underline that, in their framework, Chander and Tulkens prove that the Nash equilibrium reached is unique; this result is not extendible to all the strategic interactions, since the Nash equilibrium might not be unique, and, furthermore, different equilibria lead to different payoffs for the players.

Formally:

$$v_\gamma(S) = \max_{x_S \in X_S} \pi_S\left(x_S, x^*_{N\setminus S}\right)$$

where $x^*_{N\setminus S}$ is the Nash equilibrium of the non-cooperative game above mentioned.

CONCLUDING REMARKS

With increased competition over natural resources and environmental amenities, decision makers face strategic decisions in various management and use aspects. Cooperation approaches have been used in many cases, and were proven to be useful under certain conditions. In this review we provide information on various Cooperative Game Theory (CGT) concepts, their theoretical basis and application difficulties.

Several points have to be highlighted. CGT focuses on answering questions such as which coalitions can be formed? And how can the coalitional gains be divided in order to secure a sustainable agreement? In particular, CGT favors solutions that include all possible players (Grand Coalition), and thus most CGT solution concepts refer to the Grand Coalition. While this may not be always a desired or even feasible goal, in relatively small groups of interested parties (players) the coalitional transaction costs are relatively low and could be ignored. Thus, for relatively small environments, the promotion of the Grand Coalition is desirable.

Another important aspect associated with the solution concepts of CGT is the equitable and fair sharing of the cooperation gains. Equity is something dealt with in everyday life. It is implicitly on the agenda of agencies that are responsible to setting rules for the allocation of natural resources among users. Thus, equity should be viewed in a comprehensive way, that is, social justice: a proper distribution of resources, welfare, rights, duties, opportunities. Equity can also be viewed in a narrow framework, for example, how to solve everyday distributive problems. While this second view is the one addressed more frequently by CGT, it is still able at providing a wider look at such issues.

Acknowledging the shortcoming of CGT is important in appreciating its ability. As an axiomatic approach it has two weaknesses: first, the axioms, reasonable by themselves, may lead to "impossibility theorems"; second, the axiomatic method may result in a solution that is way too far from the practical problem dealt with: the perceived equity always depends on the particulars of the case. The various approaches reviewed in this paper and the examples used to demonstrate their application were selected to reflect both the opportunity and the difficulties embedded in CGT.

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