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INVITED PAPER

Transversality of the Shapley value

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Abstract A few applications of the Shapley value are described. The main choice criterion is to look at quite diversified fields, to appreciate how wide is the terrain that has been explored and colonized using this and related tools.

Keywords Coalitional game · Shapley value · Applied game theory · Axiomatizations · Game practice

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1 Introduction

The Shapley value was introduced[1] in 1953. Seen in retrospect, a great year for cooperative games, since in that year also the core appeared (Gillies 1953): the two solution concepts most widely studied and used.

[1]Created or discovered? A long lasting debate on mathematical research.

This invited paper is discussed in the comments available at:
http://dx.doi.org/10.1007/s11750-008-0045-4, http://dx.doi.org/10.1007/s11750-008-0046-3,
http://dx.doi.org/10.1007/s11750-008-0047-2, http://dx.doi.org/10.1007/s11750-008-0048-1,
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The title is inspired by a tutorial that one of the authors planned to deliver at the 7th meeting on Game Theory and Practice (Montreal, 2007), but was unable to do it for personal reasons. Thanks to Georges Zaccour whose invitation sparked the present survey.

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The Shapley value addresses a problem:

How to convert information about the worth that subsets of the player set can achieve, into a personal attribution (of payoff) to each of the players?

Shapley proposes an answer to this question, which is based on the idea of defining a "value" for each player involved in the game, so that players can evaluate ex-ante the convenience to participate. It is clear that there is an immediate connection of this idea with the most well-known concept of a solution at that time: the value for a zero-sum game, whose existence was proved in 1928 by von Neumann (1928), in the so-called "minimax theorem".

The approach followed by Shapley is a clever one: to provide a set of properties that a "conversion" as described above should satisfy. In other words, he uses the so-called "axiomatic approach", that proved to be so powerful just a few years before, employed by Arrow (1951 the "dictator" theorem) and by Nash (1950 the Nash bargaining solution).

Shapley succeeded in providing three conditions on the transformation from a TU-game into an allocation that can be fairly said to be natural. Actually, two of them can be considered quite compelling, from the point of view of the standard interpretation of cooperative games. The last one, which requires additivity for the transformation, is more debatable, for sure. Not incidentally, quite similar remarks can be made about the role that the "Independence of Irrelevant Alternatives" plays (in different settings) in the approach by Arrow and Nash. It is not by chance that these axioms have a key role in allowing to extend the "solution" from a small set of situations to a much broader and more interesting one.

In the fifty years elapsed since it appeared, Shapley value has shown an amazing vitality, staying in the foreground, and prompting:

- Applications to quite diverse fields (this will be the main focus of this survey);
- Introduction of new theoretical approaches to the Shapley value;
- A lot of extensions and generalizations, but also of "particularizations", that is, restrictions to smaller classes of games, of special interest for specific applications.

It is worth mentioning also that the Shapley value is used both as a *normative* tool and as a *descriptive* tool, quite similarly to what happens for the Nash bargaining solution.

So, the literature about the Shapley value is quite large, and we refer the reader, who would like to have an idea of the kind of the results that are available, to some dedicated sources or surveys, like: Roth (1988a) and six chapters in the 3rd volume of the Handbook of Game Theory (Aumann and Hart 2002): Chap. 53 by Winter (2002), 54 by Monderer and Samet (2002), 55 by McLean (2002), 56 by Neyman (2002), 57 by Aumann and Hart (2002), and 58 by Mertens (2002). See also the on-line bibliography Hart (2006).

From a broader point of view, we mention some books that offer a general introduction to game theory and, especially, to cooperative games: Owen (1995), Myerson (1991), and Osborne and Rubinstein (1994).

The aim of this contribution is not to provide a survey of the Shapley value, its theoretical developments, and the bulk of its applications. We shall look at some *specific* *cases of application*, trying to emphasize the *diversity* of the fields and disciplines to which the Shapley value has been applied. Whenever appropriate, we shall also emphasize the variants that have been introduced to better fit the application at hand. We shall discuss applications to the following fields: cost allocation (especially, the costs of infrastructures), social networks, water-focused issues, biology, reliability theory, belief formation.

We feel the need to stress that the topics chosen are not meant to be "the most important", or the like. We used some kind of "bounded rationality" criterion, i.e., chose some examples, characterized by the fact that they are significantly distant each other, with the hidden goal to encourage the search for applications that could extend the "scope" of game-theoretical methods.

The structure of this contribution is as follows: after a section devoted to establish notations, terminology, and definitions, the Shapley value is introduced in Sect. 3. Then, Sect. 4 deals with some reformulations of the original approach to the Shapley value, while Sect. 5 offers some of the extensions, generalizations, and particularizations. The remaining sections contain the discussion of the special topics outlined above. A very short concluding section precedes the list of references.

2 Notations

Given a set N, the set of its subsets will be denoted by $\mathcal{P}(N)$, while $\mathcal{P}_{\blacksquare}(N)$ refers to all *nonempty* subsets, and $\mathcal{P}_2(N)$ denotes the set of all subsets of N with cardinality 2; $R \subseteq S$ means that R is a subset of S, while the notation $R \subsetneq S$ means $R \subseteq S$ and $R \neq S$.

A TU-game[2] in characteristic form[3] is (N, v), where:

- N is a finite set (whose elements are usually said to be the "players")
- $v : \mathcal{P}(N) \to \mathbb{R}$ is a map, with $v(\emptyset) = 0$.

As it is customary in this setting, we shall be quite loose in the notation used to identify sets, to avoid to be too cumbersome. We shall use $v(i)$ instead of $v(\{i\})$, $v(ij)$ instead of $v(\{i, j\})$, $v(S \cap i)$ instead of $v(S \cap \{i\})$, and so on. We shall also use in a systematic way lowercase letters to indicate the number of elements in a set: in particular, given a coalition S, its cardinality will be referred to as s.

We shall say that a TU-game is:

- *Superadditive*, if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ s.t. $S \cap T = \emptyset$;
- *Cohesive*, if $v(N) \geq \sum_{k=1}^{m} v(S_k)$ for any partition $\{S_1, \dots, S_m\}$ of N;
- *Convex*, if $v(S \cup i) - v(S) \geq v(T \cup i) - v(T)$ for all $S, T \subseteq N$ s.t. $S \supseteq T$.

The class of all TU-games with player set N will be denoted by $\mathcal{G}(N)$, while $\mathcal{SG}(N)$ denotes the set of superadditive games.

Often, instead of looking at the worth of coalitions, one focuses on the *costs* attributed (or due) to the coalitions. We shall usually employ the notation $c(S)$ when

[2] Also said: "side-payment game", or "coalitional game".

[3] Or in "characteristic function form", or also "in coalitional form".