Game Theory

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Capitolo 1

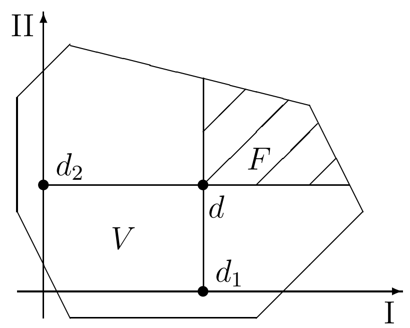
Cooperative Games

1.1 The Bargaining Problem

The two player bargaining problem was introduced by Nash in 1950.

The cooperative approach allows the players to agree on any pair in the strategy space $X \times Y$.
Under suitable hypotheses on the compactness of the strategy space (e.g. a simplex) and on the behavior of the utility functions (e.g. linearity), the image in the utility space $I \times II$ is a convex closed set V.

For each player i a reference value d_i is considered, e.g. the maxmin solution, the Nash solution or any other; the pair $d = (d_1, d_2)$ represents the payoff of the two players when they don't reach an agreement. The subset of $F = V \cup \{(x_1, x_2)|x_1 \geq d_1, x_2 \geq d_2\}$ represents the set of payoffs that the players may reach after a bargaining process; F is close, convex, bounded and non-empty.



Definition 1.1.1 *A two players bargaining problem is a pair (F, d) where $F \subset \mathbb{R}^2$ is the* feasibility set *that results close, convex, bounded and non-empty and $d = (d_1, d_2) \in \mathbb{R}^2$ is the* disagreement point*.*

The most interesting part of the problem is when the payoff of one player increases only if the payoff of the other decreases. This matter can be easily checked considering the Pareto boundary of F (efficient players).

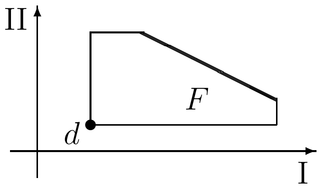
1.1.1 Nash Axiomatic Solution (1950)

A solution, or solution rule, $\Phi(F, d)$ for a two players bargaining problem $(F, d) \in \mathcal{C}$, where $\mathcal{C}$ represents the set of all two players bargaining problems, is a function $\Phi : \mathcal{C} \to \mathbb{R}^2$ such that $\Phi(F, d) \in F$ and verifies the following *Nash axioms*:

1. Strong efficiency

The solution belongs to the strong Pareto boundary, i.e. it cannot be improved by a player unless a reduction for the other player:

$$x \in F, x \geq \Phi(F, d) \Rightarrow x = \Phi(F, d)$$



2. Individual rationality

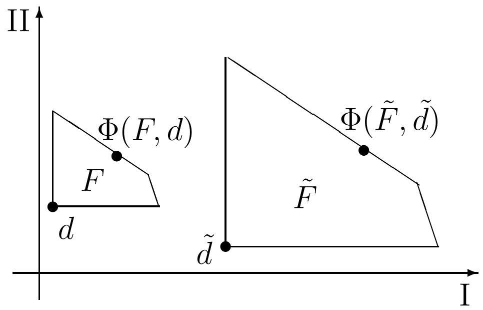
$$\Phi(F, d) \geq d$$

with the classical order defined on $\mathbb{R}^2$.

3. Scale covariance

The solution is invariant under linear trasformations, i.e. $\forall\ \lambda_1, \lambda_2 \in \mathbb{R}_>, \forall\ \mu_1, \mu_2 \in \mathbb{R}$ let $\tilde{F} = \{(\lambda_1 x_1 + \mu_1, \lambda_2 x_2 + \mu_2) \mid (x_1, x_2) \in F\}$ and $\tilde{d} = (\lambda_1 d_1 + \mu_1, \lambda_2 d_2 + \mu_2)$ then:

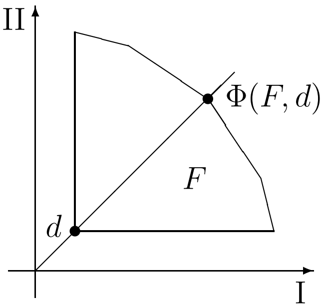
$$\Phi(\tilde{F}, \tilde{d}) = (\lambda_1 \Phi_1(F, d) + \mu_1, \lambda_2 \Phi_2(F, d) + \mu_2)$$



4. Symmetry

If F is symmetric for the two players, i.e. both of them can obtain the same payoff, i.e. $(a, b) \in F \iff (b, a) \in F$ and $d_1 = d_2$ then:

$$\Phi_1(F, d) = \Phi_2(F, d)$$

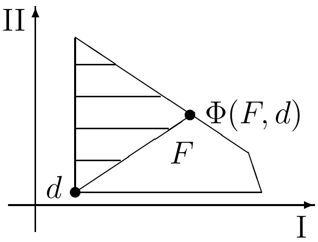


5. Indipendence from irrelevant alternatives

Debatable axiom. Deleting a subset of F that does not include the disagreement point d and the solution $\Phi(F,d)$, the solution remain the same, i.e.:

$$d, \Phi(F,d) \in G \subset F, (G,d) \in \mathcal{C} \Rightarrow \Phi(G,d) = \Phi(F,d)$$

In the example in the following picture, the first player may or may not accept the previous solution.



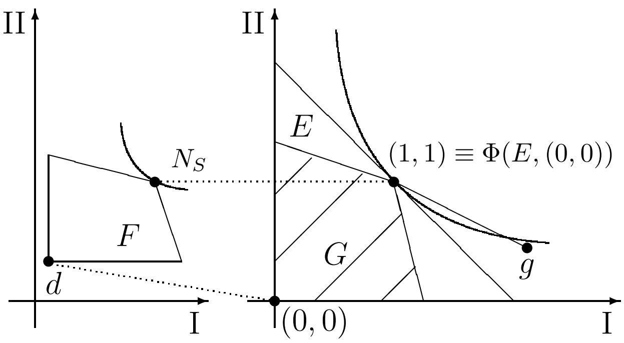
Theorem 1.1.1 *There exists a unique function $\Phi : \mathcal{C} \to \mathbb{R}^2$ satisfying the previous axioms; this function associates to each problem (F,d) the point that maximizes the Nash product:*

$$\Phi(F,d) = argmax\ \{(x_1 - d_1)(x_2 - d_2) | x \in F\} = N_S$$

Proof

We may construct a suitable function Φ, starting from the known elements. For a given (F,d), there exists a unique point $(x_1, x_2) \in F$ such that $(x_1 - d_1)(x_2 - d_2)$ is maximal. By axiom 3 it is possible to define a linear trasformation from F to G such that:

$$\begin{array}{rcll} d & \to & (0,0) & \quad (*) \\ (x_1,x_2) & \to & (1,1) & \quad (*) \end{array}$$



The triangle E is symmetric, so by axiom 4 the solution of the bargaining problem $(E,(0,0))$ is the point (1, 1); by axiom 5 the point (1, 1) is also the solution of the problem $(G,(0,0))$. Now it suffices to check that $G \subset E$; let $g \in G$ be a point such that $g \notin E$. By the convexity of G the segment $[(1,1),g] \subset G$ includes also points where the Nash product assumes values larger than in (1, 1). Contraddiction. ♣

Remark 1.1.1

- *If the payoff in F of a player is constant also the Nash product is constant.*
- *In the above discussion, some hypotheses were stated:*
  1. *the influence of the disagreement point on the solution;*
  2. *the agents are decision makers á la von Neumann - Morgenstern.*

  If the hypotheses do not hold, the final solution can be different; for example given two equivalent bargaining problems, in one of the two one of the agents could behave differently, leading to different results.

The bargaining problem inspired other solution concepts for NTU games and TU games; among these last there are the Bargaining set (Aumann e Maschler, 1964), the Kernel (Davis e Maschler, 1965) and the Nucleolus (Schmeidler, 1969).

1.1.2 Another solution

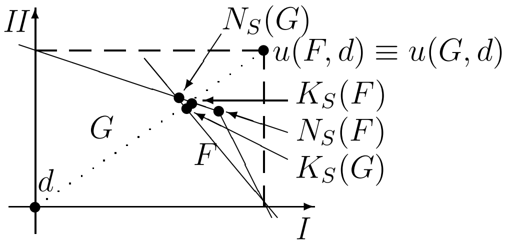
Axiom 5 was revised by Kalai-Smorodinsky (1975). They proposed the following:
<u>5'. Individual Monotonicity</u>
Let $d \in G \subset F$.
If $u_1(G,d) = u_1(F,d)$ then $\Phi_2(G,d) \leq \Phi_2(F,d)$ and if $u_2(G,d) = u_2(F,d)$ then $\Phi_1(G,d) \leq \Phi_1(F,d)$, where $u(F,d)$ is the *utopia* point of the problem (F,d), i.e. $u_i(F,d) = max\ \{x_i \mid x \in F\}, i = 1,2$.

Kalai and Smorodinsky proposed the following solution:

$$K_S = argmax\ \left\{ \frac{x_1 - d_1}{u_1(F,d) - d_1} \;\middle|\; x \in F, \frac{x_1 - d_1}{u_1(F,d) - d_1} = \frac{x_2 - d_2}{u_2(F,d) - d_2} \right\}$$

Axiom 5' does not hold for the Nash solution, as in the following example.

Example 1.1.1 (Nash and Kalai-Smorodinsky Solutions) *Consider the following situation:*



Reducing F to G, the utopia point is the same, nevertheless $N_{S2}(G) > N_{S2}(F)$ while $K_{S2}(G) < K_{S2}(F)$. ♢

1.2 Cooperative Games

In a strategically interactive situation, the agents may have a common aim, so that they can be interested in cooperating, in order to improve their utility.

Definition 1.2.1 *Let N be the player set, each subset S of N is a* coalition*. If $S = N$ it is the* grand coalition*.*

The agents that form a coalition cooperate in the game.
The cooperation requires:

- the possibility of agreements, i.e. there do not exist antitrust rules;
- the possibility of forcing the respect of the agreements, i.e. there exists a superpartes authority that all the agents accept.

In general, cooperative games are divided in two classes, depending on the possible division of the total payoff, after the cooperation:

- Cooperative games without transferable utility (NTU-Games): the players receive the payoff according to the strategy profile they agreed upon.
- Cooperative games with transferable utility (TU-Games): the players may share the total payoff generated by the strategy profile they agreed upon.

TU-Games are a special case of NTU-Games.

More precisely, a TU-Game has to satisfy the following three additional hypotheses:

- it is possible to transfer the utility;
- there exists a common exchange tool, e.g. the money, that allows to transfer the utility (in a material sense);

- the utility functions of the players must be equivalent, e.g. they can be linear in the amount of money.

Remark 1.2.1

- *The decision on the sharing of the total payoff in a TU-Game is part of the binding agreement.*

Example 1.2.1 (Simple coalition) *Given three players I, II, III, if two of them agree, forming a coalition, then the third player has to give to each of them a unit of money, otherwise the three players get nothing. The corresponding payoffs are:*

$(1, 1, -2)$	*if I and II coalize*
$(1, -2, 1)$	*if I and III coalize*
$(-2, 1, 1)$	*if II and III coalize*
$(0, 0, 0)$	*otherwise*

Supposing that the payoffs of the coalition (II, III) are $(-2.0, 1.1, 0.9)$*, player II is in a weaker position, as player III prefers to join with player I, unless player II may transfer part of his payoff to player III.* $\diamondsuit$

Definition 1.2.2

- *The* characteristic function *of a n-person game is denoted by v (in a NTU-Game the notation is V and the definition if more complex):*

$$v : \wp(N) \to \mathbb{R} \text{ with } v(\emptyset) = 0$$

- *If for every pair of disjoint coalitions S and T, if $v(S \cup T) = v(S) + v(T)$ the function v is additive; if $v(S \cup T) \geq v(S) + v(T)$ v is superadditive; if $v(S \cup T) \leq v(s) + v(T)$ v is subadditive.*

In other words v assigns to the coalition S the maximal payoff, independently from the behavior of the other players.

Remark 1.2.2

- *In general, the characteristic function is enough for describing the game, consequently they may be identified.*

A game defined as a pair $G = (N, v)$ is represented in*characteristic form* or *coalitional form.*

If the characteristic function is additive or superadditive or subadditive also the game is said *additive* or *superadditive* or *subadditive.* If $v(S) + v(N \setminus S) = v(N)$ for each coalition S it is a *constant sum game.*

If the payoffs of the players are negative, i.e. the coalitions have to pay or have a loss, it is convenient to represent the game as a *cost game* (N, c), where $c = -v$.

1.2.1 Characteristic Function for a TU Game

The characteristic function of a TU Game may be obtained starting from the strategic or normal form of the game, taking into account the different possible total payoffs of a given coalition $S \subseteq N$. Unfortunately the resulting value of $v(S)$ is not univocally determined:

$$v'(S) = \max_{\sigma_S \in \Sigma_S} \min_{\sigma_{N\setminus S} \in \Sigma_{N\setminus S}} \left\{ \sum_{i \in S} u_i(\sigma_S, \sigma_{N\setminus S}) \right\} \quad \text{(von Neumann-Morgenstern)}$$
$$v''(S) = \min_{\sigma_{N\setminus S} \in \Sigma_{N\setminus S}} \max_{\sigma_S \in \Sigma_S} \left\{ \sum_{i \in S} u_i(\sigma_S, \sigma_{N\setminus S}) \right\}$$

In the previous definition of $v(S)$ the two values, in general, do not coincide as $v''(S) \geq v'(S)$. Other definitions of the value of the characteristic function exist, so that the problem of determining the right value of v is a hard one.

Sometimes the features of the problem allow defining the characteristic function in an easy way, as in the following example.

Example 1.2.2 (Glove Game) *Let L and R be two disjoint sets of players, endowed with some gloves; the players in L own only left gloves, while the players in R own only right gloves. The value of a coalition depends on the number of pairs they are able to form. In general each player is endowed with a single glove. Supposing that $L = \{1, 2\}$, $R = \{3, 4\}$ and the value of each pair is 1, the resulting game is:*

$$\begin{array}{ll} N = \{1,2,3,4\} & \\ v(i) = 0 & \forall\ i \in N \\ v(12) = v(34) = 0 & \\ v(S) = 1 & \textit{if } |S| = 2 \textit{ and } S \neq \{12\}, S \neq \{34\} \textit{ or if } |S| = 3 \\ v(N) = 2 & \end{array}$$

$\diamondsuit$

Definition 1.2.3 *A game $G = (N, v)$ is said* monotonic *if $v(S) \leq v(T),\ \forall\ S \subseteq T$.*

Definition 1.2.4 *A game $G = (N, v)$ is said* convex *if one of the following equivalent conditions holds:*

- $v(S) + v(T) \leq v(S \cup T) + v(S \cap T),\ \forall\ S, T \subseteq N.$
- $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T),\ \forall\ S \subset T \subseteq N \setminus \{i\}.$

Definition 1.2.5 *A game $G = (N, v)$ is said* cohesive *if for each partition of N $\{S_1, S_2, \ldots, S_k\}$:*

$$\sum_{i=1,\ldots,k} v(S_i) \leq v(N)$$

Remark 1.2.3

- *The definition of monotonicity is independent from the cardinality of the coalitions.*

- *Cohesiveness is weaker than superadditivity and represents the "interest" of the players in forming the grand coalition, rather than several subcoalitions. This property is relevant as many solution concepts consist in dividing the value of the grand coalition among all the players.*

The solutions of a TU Game may be divided in two groups:

- *set solutions*: a set of payoff vectors is associated to the player set;

- *point solutions*: a single payoff vector is determined; this kind of solutions is closer to the classical idea of solution of a problem.

Capitolo 2

Set Solutions for TU Games

2.1 Imputations

The first idea to determine the payoff of each player may consist in equally sharing the value of the game among the players, taking in no account the contribution of each player; another possibility could be to define a subgame restricted to the players of each coalition. A different approach is rooted in the analysis of the role of the players, leading to various solution concepts.

Definition 2.1.1 *Given a game* $G = (N, v)$ *an* imputation *is a vector* $x = (x_1, x_2, \ldots, x_n)$ *such that:*

$$\sum_{i \in N} x_i = v(N) \qquad \textit{efficiency}$$
$$x_i \geq v(i); i = 1, \ldots, n \qquad \textit{individual rationality}$$

For a cost game $G = (N, c)$ *individual rationality is expressed as* $x_i \leq c(i), \forall\ i \in N$.

The set of all the imputations is denoted by $E(v)$.

Definition 2.1.2 *Given a game* $G = (N, v)$*, if* $\sum_{i \in N} v(i) = v(N)$ *holds then the unique element of* $E(v)$ *is* $x = (v(1), v(2), \ldots, v(n))$*; when* $E(v)$ *contains more than one vector the game is said* essenzial.

Individual rationality means that each player receives at least the payoff he can get standing alone. An imputation is the first step toward a solution concept that respects the role of the players. On the other hand for an essential game there exist many imputation vectors, so again we have the problem of choosing a solution. In fact by efficiency, given two different imputations x and y there exists at least one player k such that $x_k > y_k$ and at least one player h such that $x_h < y_h$.

2.2 Core

The most interesting set solution for many classes of games is the core, introduced by Gillies (1953 and 1959). The starting point is to extend the individual rationality hypothesis to all the coalitions:

$$x(S) \geq v(S) \quad S \subset N \qquad \text{coalitional rationality}$$

where $x(S) = \sum\limits_{i \in S} x_i$.

For a cost game $G = (N, c)$ coalitional rationality is expressed as $x(S) \leq c(S), \forall\ S \subset N$.

Definition 2.2.1 *Given a game $G = (N, v)$, the core is the set:*

$$C(v) = \{x \in E(v) | x(S) \geq v(S), \forall\ S \subset N\}$$

Remark 2.2.1

- *The core may be empty, as for an essential constant sum game.*
- *The core is useful to select which solutions should not be chosen (those not belonging to the core) when the core is non empty. The emptyness of the core does not imply that the grand coalition does not form, but gives clues on its low stability.*

2.3 Examples

This section is devoted to two examples of the characteristics of the core.

2.3.1 Glove Game

Referring to the Example 1.2.2, the core is the set:

$$C(v) = \{(\alpha, \alpha, 1 - \alpha, 1 - \alpha) s.t. 0 \leq \alpha \leq 1\}$$

In general, for a glove game with left player set $L = \{1, ..., n_l\}$ and right player set $R = \{1, ..., n_r\}$, if $n_l = n_r$ the core is the set:

$$C(v) = \{(\alpha, ..., \alpha, 1 - \alpha, ..., 1 - \alpha) s.t. 0 \leq \alpha \leq 1\}$$

while if $n_l < n_r$ the core is the set:

$$C(v) = \{(\underbrace{1, ..., 1}_{1,...,n_l}, \underbrace{0, ..., 0}_{1,...,n_r})\}$$

and if $n_l > n_r$ the core is the set:

$$C(v) = \{(\underbrace{0, ..., 0}_{1,...,n_l}, \underbrace{1, ..., 1}_{1,...,n_r})\}$$

Remark 2.3.1

- *The core of this game represents the situation in which a shortage of complementary goods results in an advantage for the owners of more required goods.*

2.3.2 Assignment Game

The agents are divided in two groups, sellers and buyers; each seller has just one object to sell, whose evaluation is known to him; each buyer may purchase a single object and knows the evaluation of each object. A seller with more then one object or a buyer interested in several objects may be represented with identical copies of the players, in particular with identical evaluations of the objects.
The objects has no fixed market prices, but the selling prices depend on the evaluations and on the bargaining ability. The aim of each player is the maximal gain. Formally an assignment problem is a 4-tuple $\mathcal{A} = (N^v, N^c, A, B)$ where $N^v = \{1, ..., n^v\}$ is the seller set, $N^c = \{1, ..., n^c\}$ is the buyer set, A is a vector with a_j representing the evaluation of seller $j \in N^v$ of his own object, B is a matrix with b_{ij} corresponding to the evaluation of buyer $i \in N^c$ for the object of seller $j \in N^v$.

It is possible to associate a TU game (N, v) to an assignment problem where $N = N^v \cup N^c$ and v is defined as follows:

- if seller j^* and buyer i^* form a coalition then:

$$v(i^*j^*) = c_{i^*j^*} = \begin{cases} b_{i^*j^*} - a_{j^*} & \text{if } \ b_{i^*j^*} - a_{j^*} \geq 0 \\ 0 & \text{if } \ b_{i^*j^*} - a_{j^*} < 0 \end{cases}$$

- if a coalition S includes more buyers then sellers, let $i(j) \in S \cap N^c$ be the buyer that purchase the object of the seller $j \in S \cap N^v$, then:

$$v(S) = max \sum_{j \in S \cap N^v} c_{i(j),j}$$

- if a coalition S includes more sellers then buyers, let $j(i) \in S \cap N^v$ be the seller of the object given to buyer $i \in S \cap N^c$, then:

$$v(S) = max \sum_{i \in S \cap N^c} c_{i,j(i)}$$

The set of values c_{ij} defines the assignment problem:

$$\begin{array}{lll} max & z = \sum_{i \in N^c, j \in N^v} c_{ij} x_{ij} & \\ s.t. & \sum_{i \in N^c} x_{ij} \leq 1 & \forall\, j \in N^v \\ & \sum_{j \in N^v} x_{ij} \leq 1 & \forall\, i \in N^c \\ & x_{ij} \in \{0, 1\} & \forall\, i \in N^c, j \in N^v \end{array}$$

where $x_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ agree} \\ 0 & \text{otherwise} \end{cases}$.

The core of an assignment game contains the imputations associated to an optimal solution of the dual problem (Owen, 1975 and Shapley-Shubik, 1972):

$$\begin{array}{lll} min & w = \sum_{j \in N^v} y_j^v + \sum_{i \in N^c} y_i^c & \\ s.t. & y_j^v + y_i^c \geq c_{ij} & \forall\, j \in N^v, \forall\, i \in N^c \end{array}$$

Example 2.3.1 (Assignment Game) *Let* $N^v = \{1\}$, $N^c = \{2,3\}$, $a_1 = 10$ *and* $b_{21} = 12, b_{31} = 15$. *The characteristic function is:*

$$v(1) = v(2) = v(3) = v(23) = 0; v(12) = 2; v(13) = v(N) = 5$$

The core is the set of imputations (x_1, x_2, x_3) *with* $x_1 = \alpha, x_2 = 0, x_3 = 5 - \alpha, 2 \leq \alpha \leq 5$. *This implies that the object is not sold to player 2; the payoff of players 1 and 3 depends on the selling agreement, but the selling price is at least 12, i.e. the utility of player 1 is at least 2 units, so that player 3 is sure to exclude player 2. On the other hand the selling price cannot exceed 15 otherwise both buyers reject. Supposing that the evaluation of player 2 is* $\bar{b}_{21} = 15$ *then the unique core allocation is* $(5, 0, 0)$, *i.e. the selling price is exactly 15.* $\diamondsuit$

Remark 2.3.2

- *Example 2.3.1 allows the following economic remarks.*

  1. *The equilibrium of demand and offer says that the price will be larger than 12, otherwise there will be two buyers and one seller and smaller than 15 otherwise there will be one seller and no buyer.*

  2. *Economic laws allow a positive payoff for player 2; this situation takes into account other possible roles of the player. For example player 1 can collude with player 2: if he increases his offer the price paid by player 3 will be higher; on the other hand also player 3 can collude with player 2: if he leaves from the market the price paid to player 1 will be lower.*

Capitolo 3

Point Solutions for TU Games

This class of solutions includes the so-called *power indices* and *values*. Power indices are used in simple games in order to evaluate the relevance or power of each player; values are widely used as allocation rules. In general an axiomatic characterization of a value enables the player to emphasize the reasons that lead to adopt it as allocation rule. The most important value is due to Shapley.

3.1 Shapley Value (1953)

This value is rooted in the concept of *marginal contribution*, i.e. the variation of value of a coalition after that a player enters.

Definition 3.1.1 *Given a game $G = (N, v)$, the Shapley value is the vector $\phi(v)$ whose component ϕ_i is the average marginal contribution of player i w.r.t. all the permutations of the players:*

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi} [v(P(\pi, i)) \cup \{i\}) - v(P(\pi, i))]$$

where $n = |N|$, π is a permutation of N and $P(\pi, i)$ is the set of players preceeding i in the permutation π.

Given a TU game, the Shapley value always exists and is unique.

If the game is superadditive (subadditive for a cost game) the Shapley value is an imputation as:

$$\sum_{i \in N} \phi_i(v) = v(N)$$
$$\phi_i(v) \geq v(i) \qquad \forall i \in N$$

but not always belongs to the core, even if the core is non empty (see Example 3.1.1).

If the game is convex (concave, in the case of a cost game) the Shapley lies in the core.

Example 3.1.1 (Assignment Game) *Referring to the Example 2.3.1, where the characteristic function is* $v(1) = v(2) = v(3) = v(23) = 0; v(12) = 2; v(13) = v(123) = 5$ *the Shapley value is given by:*

	Marginal contributions		
Permutations	*Player 1*	*Player 2*	*Player 3*
1 2 3	$v(1) - v(\emptyset) = 0$	$v(12) - v(1) = 2$	$v(123) - v(12) = 3$
1 3 2	$v(1) - v(\emptyset) = 0$	$v(123) - v(13) = 0$	$v(13) - v(1) = 5$
2 1 3	$v(12) - v(2) = 2$	$v(2) - v(\emptyset) = 0$	$v(123) - v(12) = 3$
2 3 1	$v(123) - v(23) = 5$	$v(2) - v(\emptyset) = 0$	$v(23) - v(2) = 0$
3 1 2	$v(13) - v(3) = 5$	$v(123) - v(13) = 0$	$v(3) - v(\emptyset) = 0$
3 2 1	$v(123) - v(23) = 5$	$v(23) - v(3) = 0$	$v(3) - v(\emptyset) = 0$
ϕ_i	$\frac{17}{6}$	$\frac{2}{6}$	$\frac{11}{6}$

The Shapley value takes into account the behavior of player 2 given in Remark 2.3.2. $\diamondsuit$

3.1.1 Axiomatization of the Shapley Value

The axiomatic characterization of the Shapley value requires the following three axioms in addition to efficiency:

1. Symmetry
   Given a game $G = (N, v)$, if two players i, j are symmetric, i.e. $v(S \cup \{i\}) = v(S \cup \{j\}), \forall\ S \subseteq N \setminus \{i, j\}$ then $\phi_i(v) = \phi_j(v)$.

2. Dummy player
   Let i be a player that adds to each coalition only his own value $v(i)$, i.e.:
   $$v(S \cup \{i\}) = v(S) + v(i) \quad \forall\ S \subseteq N \setminus \{i\}$$
   then the Shapley value of player i coincides with his value, i.e. $\phi_i(v) = v(i)$.

3. Additivity or independence (Debatable axiom)
   Given two games with characteristic functions v and u, respectively, let $(u + v)$ the sum game defined as:
   $$(u + v)(S) = u(S) + v(S), \quad \forall\ S \subseteq N$$
   The Shapley value of the sum game is given by the sum of the Shapley values, i.e. $\phi_i(u + v) = \phi_i(u) + \phi_i(v), \forall\ i \in N$.

Example 3.1.2 (Symmetric players and dummy player) *Given the game* $G = (N, v)$ *where:*

$$N = \{1, 2, 3\}$$
$$v(1) = v(2) = v(3) = 1; v(12) = 4; v(13) = v(23) = 2; v(N) = 5$$

Players 1 and 2 are symmetric and player 3 is a dummy, so $\phi_3(v) = v(3) = 1$ *and* $\phi_1(v) = \phi_2(v) = \frac{1}{2}(v(N) - v(3)) = 2$*, then* $\phi(v) = (2, 2, 1)$. $\diamondsuit$

Remark 3.1.1

- *The axiom of symmetry can be replaced by the axiom of* anonymity*:*
  Given a game $G = (N, v)$ and a permutation π of the players, let u be the game defined as:
  $$u(\pi(S)) = v(S) \quad \forall\ S \subseteq N$$
  The Shapley value of the permutated game is given by the corresponding permutation of the Shapley value, i.e. $\phi_{\pi(i)}(u) = \phi_i(v)$.

- *Also the axiom of dummy player can be replaced by the axiom of* null player*:*
  Let $i \in N$ be such that $v(S \cup \{i\}) = v(S), \forall\ S \subseteq N \setminus \{i\}$.
  In this case $v(i) = v(\emptyset \cup \{i\}) = v(\emptyset) = 0$ and again $v(S \cup \{i\}) = v(S) + v(i), \forall\ S \subseteq N \setminus \{i\}$.

3.1.2 A Real Application of the Shapley Value

Example 3.1.3 (EU Council 1958-1973) *The Shapley enlights a worm in the weights assigned to the EU countries in the Council. In 1958, the majority quota was 12 on 17 ($\approx$ 70%) while in 1973 it was 41 on 58 ($\approx$ 70%).*

		1958			*1973*	
Countries	*Weight*	*%*	*Shapley*	*Weight*	*%*	*Shapley*
France	*4*	*23.53*	*0.233*	*10*	*17.24*	*0.179*
Germany	*4*	*23.53*	*0.233*	*10*	*17.24*	*0.179*
Italy	*4*	*23.53*	*0.233*	*10*	*17.24*	*0.179*
Belgium	*2*	*11.76*	*0.150*	*5*	*8.62*	*0.081*
The Netherlands	*2*	*11.76*	*0.150*	*5*	*8.62*	*0.081*
Luxemburg	*1*	*5.88*	**0.000**	*2*	*3.45*	*0.010*
United Kingdom	-	-	-	*10*	*17.24*	*0.179*
Denmark	-	-	-	*3*	*5.17*	*0.057*
Ireland	-	-	-	*3*	*5.17*	*0.057*
Total	*17*	*100.00*	*1.000*	*58*	*100.00*	*1.000*

Luxemburg reduced its percentual weight, but was no longer a dummy player. $\diamondsuit$

Capitolo 4

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