Introduction to Game Theory and Applications

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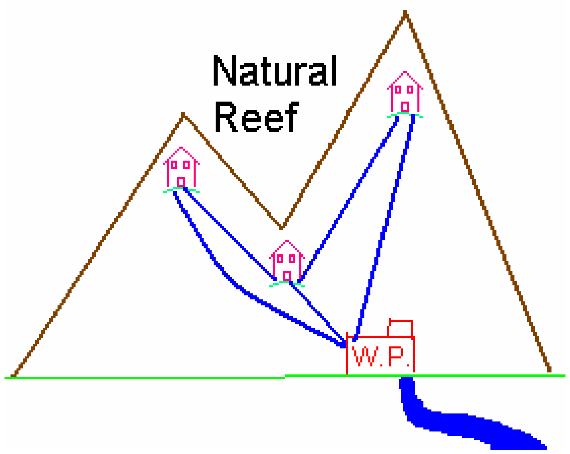
Connection situations...



- A group of agents whose houses on the mountain are not yet connected to a water purifier;
- For each agent it is sufficient, <u>but not necessary</u>, to be autonomously connected with the water purifier;
- Agents can connect also via others;
- To construct a pipe is costly.

Why a TU-game?

- **Working together**, players can realize extra savings or decrease costs, with respect to the situation where each player optimizes individually
- The new problem is: **how to divide the total cost**?

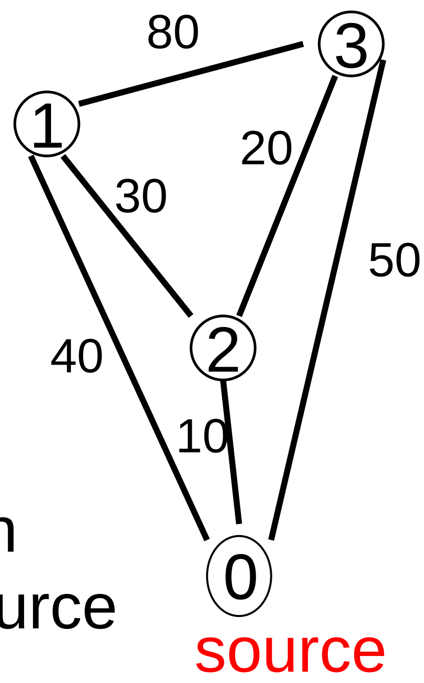
Problems on connection situations

- **fixed tree games**, games defined on problems where previously built networks must be maintained
- **minimum cost spanning tree (mcst) games,** where an optimal connection network must still be constructed.

Minimum Cost Spanning Tree Situation

Consider a complete weighted graph

- – whose vertices represent agents
- – vertex 0 is the source
- – edges represent connections between agents or between an agent and the source
- – numbers close to edges are connection costs



Minimum cost spanning tree (mcst) problem

Optimization problem:

How to connect each node to the source 0 in such a way that the cost of construction of a <u>spanning network </u>(which connects every node directly or indirectly to the source 0) <u>is minimum</u>?

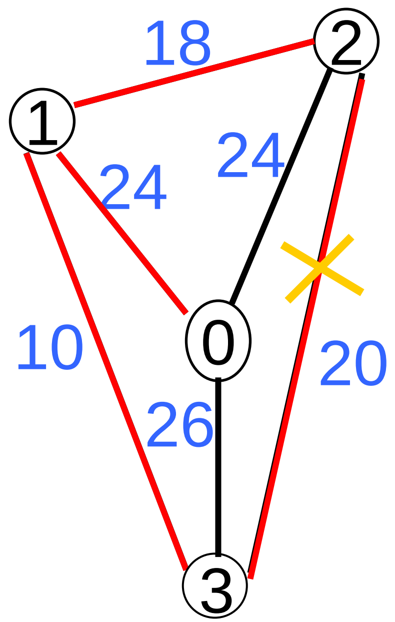
Example

N={1,2,3}

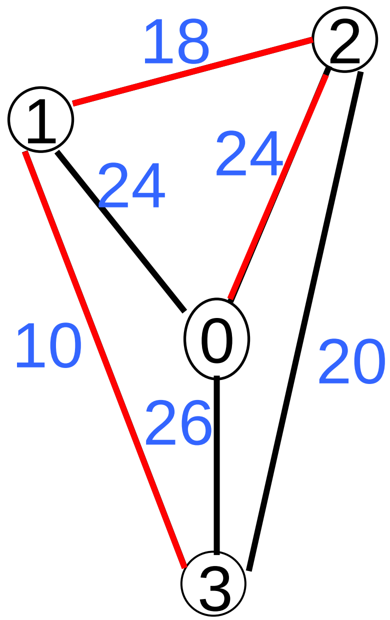
$E_{N'}$={{1,0},{2,0},{2,1},{3,0},{3,1},{3,2}}

cost function shown on graphs

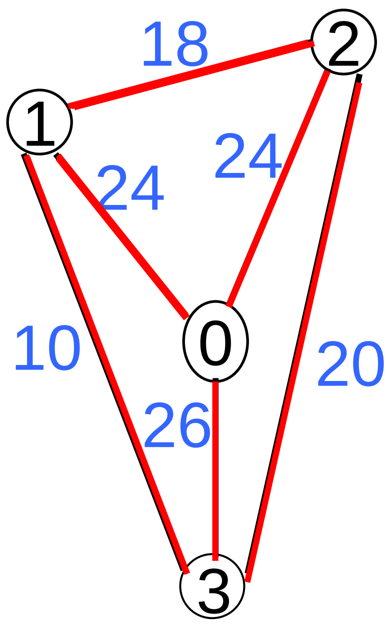
Kruskal algorithm



Prim algorithm



Example: The cost game ({1,2,3},c) is defined on the following connection situation:



c(1)=24

c(2)=24

c(3)=26

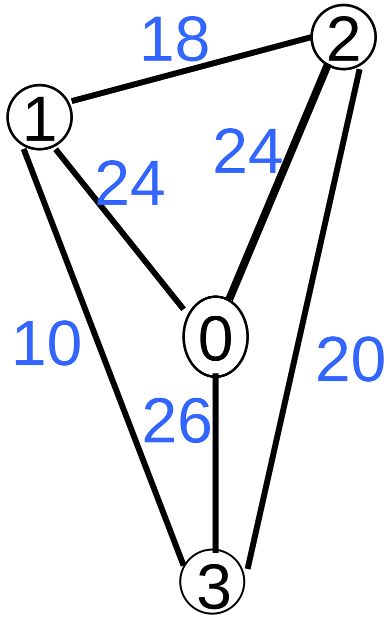
c(1,3)=34

c(1,2)=42

c(2,3)=44

c(1,2,3)=52

Example: The cost game ({1,2,3},c) is defined on the following connection situation:



c(1)=24

c(2)=24

c(3)=26

c(1,3)=34

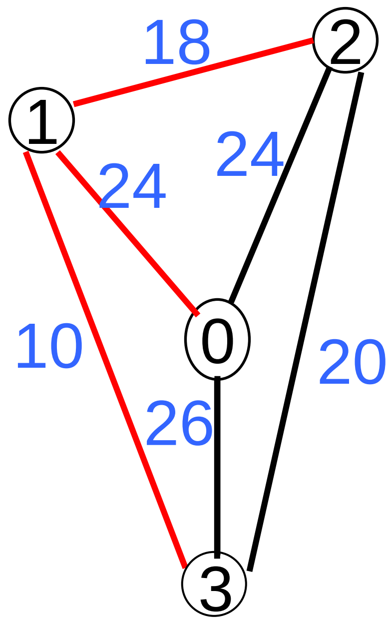
c(1,2)=42

c(2,3)=44

c(1,2,3)=52

The game ({1,2,3}, c) is said mcst game (Bird (1976))

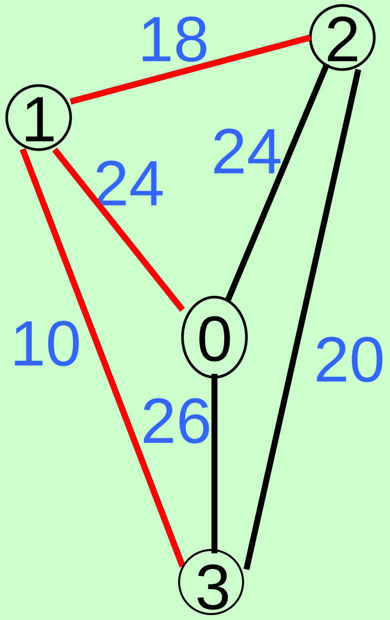
How to divide the total cost? (Bird 1976)



- The predecessor of 1 is 0: the Bird allocation gives to player 1 the cost of {0,1}.
- The predecessor of 2 is 1: the Bird allocation gives to player 2 the cost of {1,2};
- The predecessor of 3 is 1: the Bird allocation gives to player 3 the cost of {1,3}.

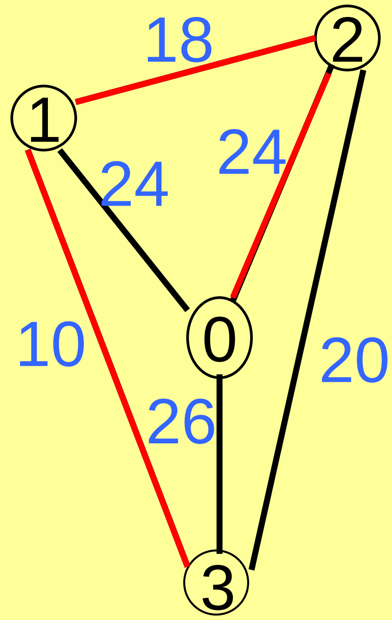
$w(\Gamma)=52$

Bird allocation w.r.t. to Γ, $(x_1, x_2, x_3)=(24, 18, 10)$ is in the core of ({1,2,3},c).



The Bird allocation w.r.t .this mcst is

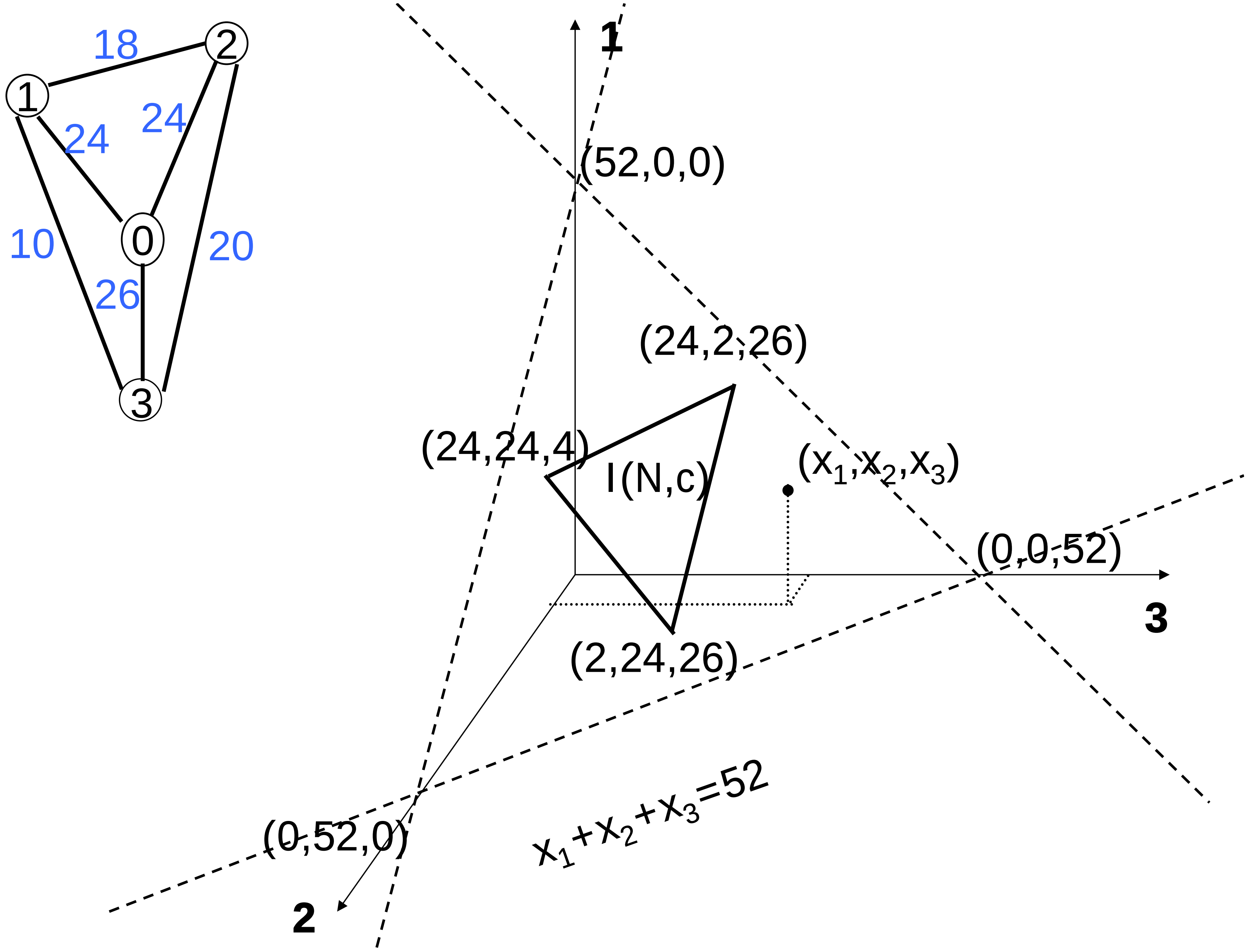
$(x_1, x_2, x_3)=(24, 18 ,10)$



The Bird allocation w.r.t. this mcst is

$(x_1, x_2, x_3)=(18, 24 ,10)$

Both allocations belong to the core of the mcst game (and also their convex combination).



18
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24
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20
26
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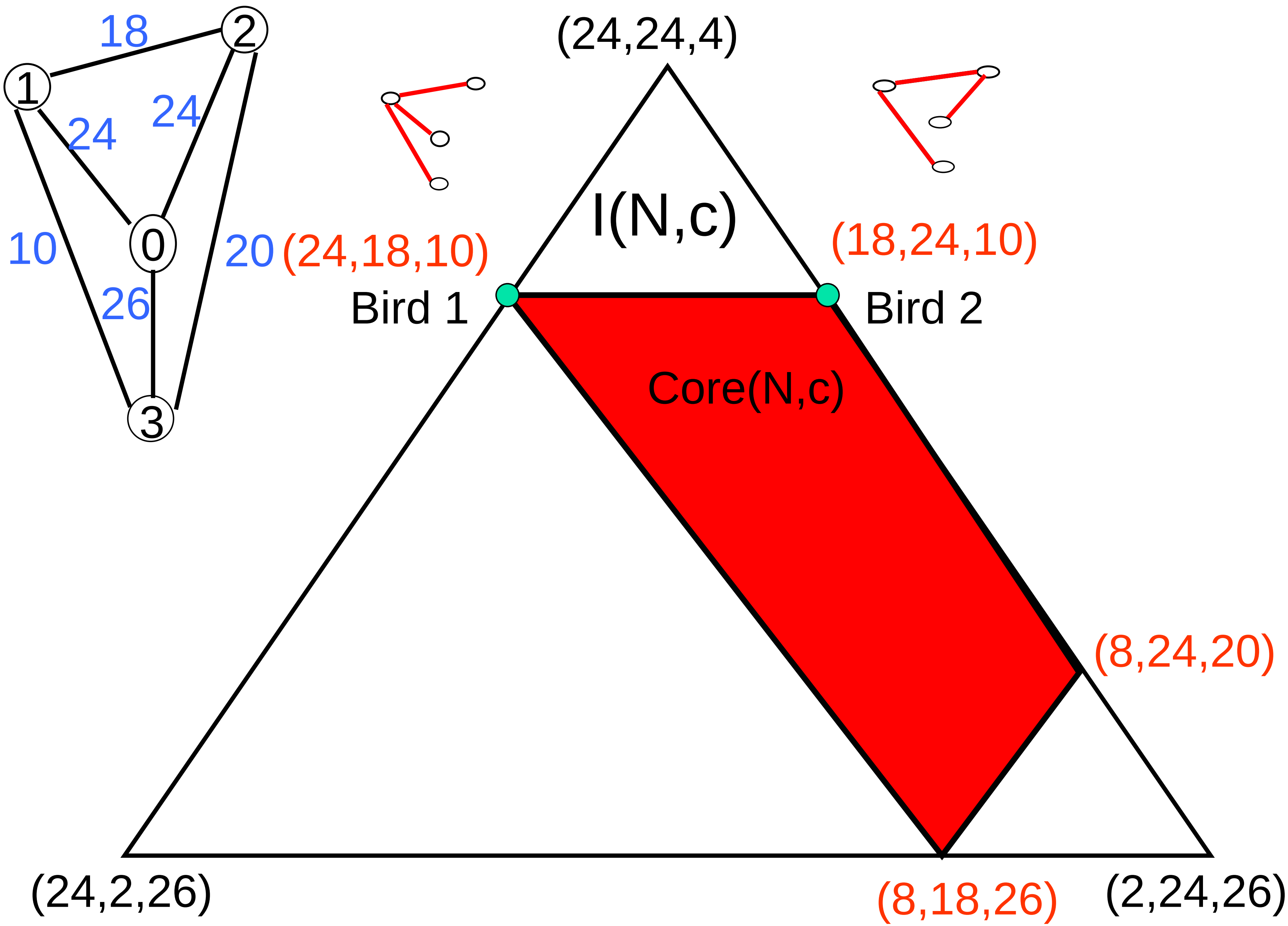
2
3
1
2
3
(52,0,0)
(24,2,26)
(24,24,4)
I(N,c)
(x_1,x_2,x_3)
(0,0,52)
(2,24,26)
(0,52,0)
$x_1+x_2+x_3=52$



18
24
24
10
20
26
1
2
0
3
(24,24,4)
I(N,c)
(24,18,10)
(18,24,10)
Bird 1
Bird 2
Core(N,c)
(8,24,20)
(24,2,26)
(8,18,26)
(2,24,26)

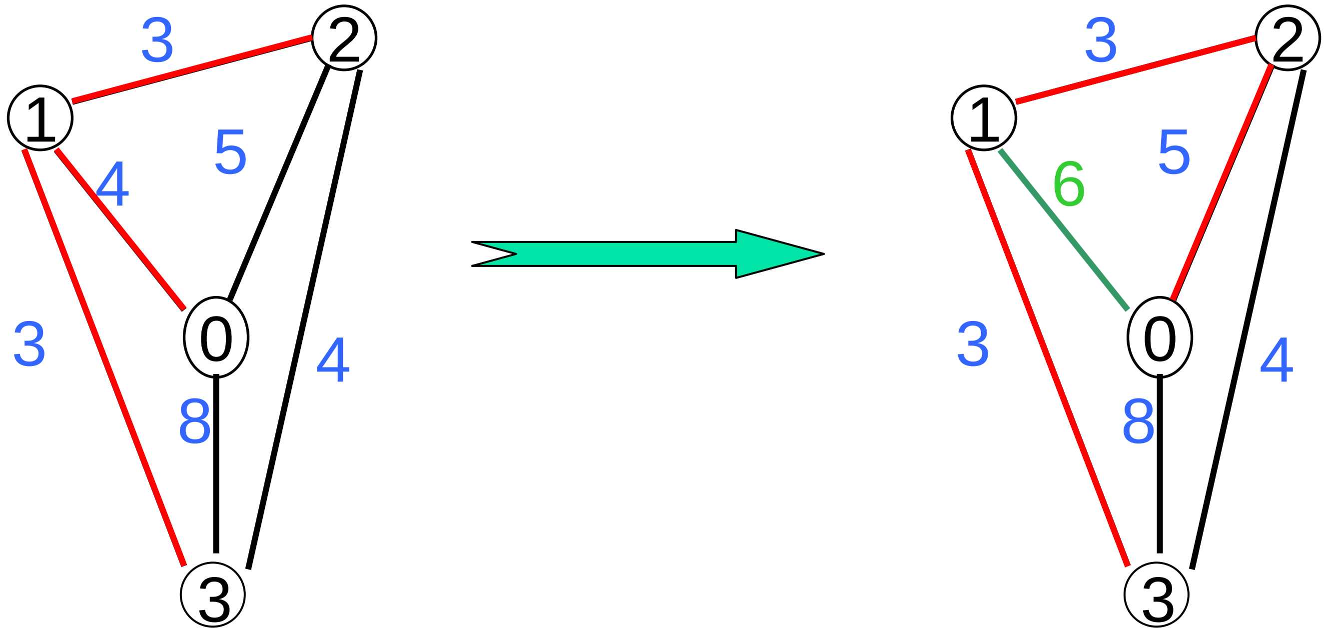
Bird allocation rule

- It always provides an allocation (given a connection situation).
- In general, not a unique allocation (each mcst determines a Bird allocation...).
- Bird allocations are in the core of mcst games (extreme points)

What happens when the structure of the network changes?

- Imagine to use a certain rule to allocate costs.
  - The cost of edges may increase: if the cost of an edge increases, nobody should be better off, according to such a rule (*cost monotonicity)*;
  - One or more players may leave the connection situation: nobody of the remaining players should be better off (*population monotonicity*).

Cost monotonicity: Bird allocation behaviour

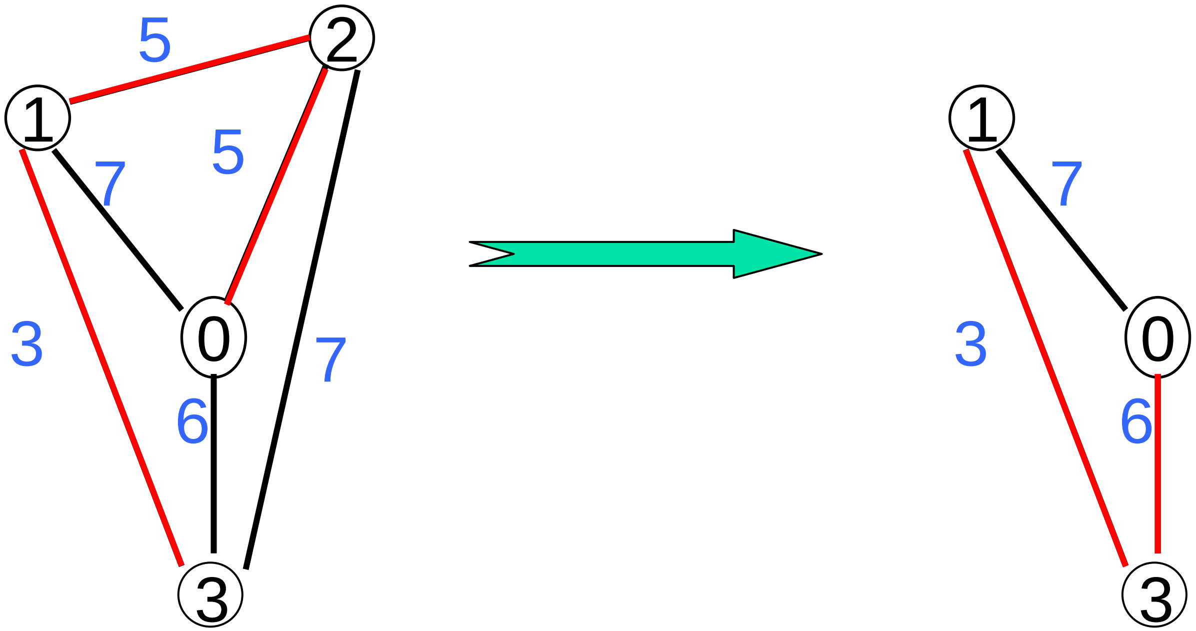


Bird allocation: (4, 3 ,3)

Bird allocation: (3, 5 ,3)

Bird rule does not satisfy cost monotonicity.

Population monotonicity: Bird allocation behaviour

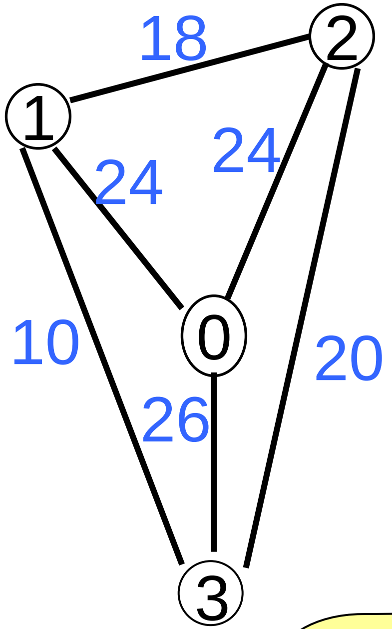


Bird allocation: (5, 5 ,3)

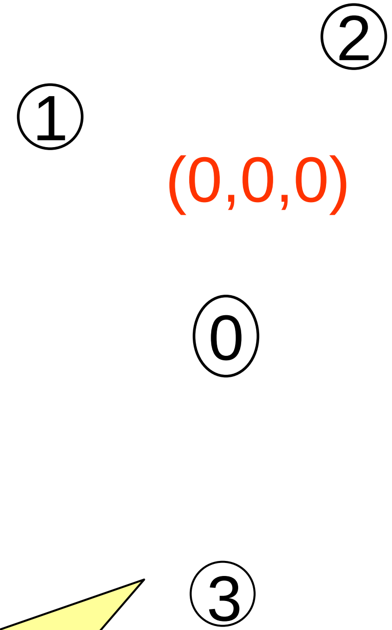
Bird allocation: (3, * ,6)

Bird rule does not satisfy population monotonicity

P-value: Feltkamp (1994), Branzei et al. (2004), Moretti (2008)

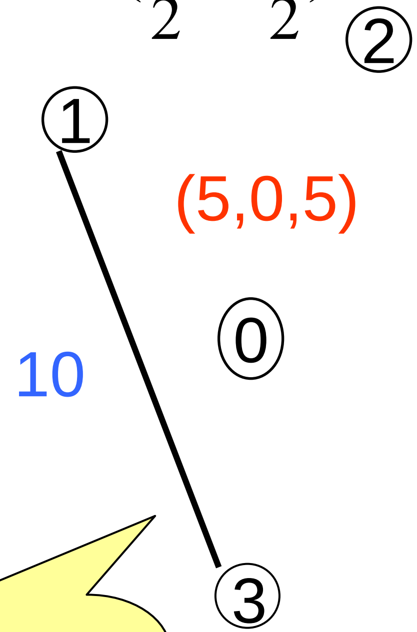


$$b^{\sigma,0} = (1,1,1)^t$$

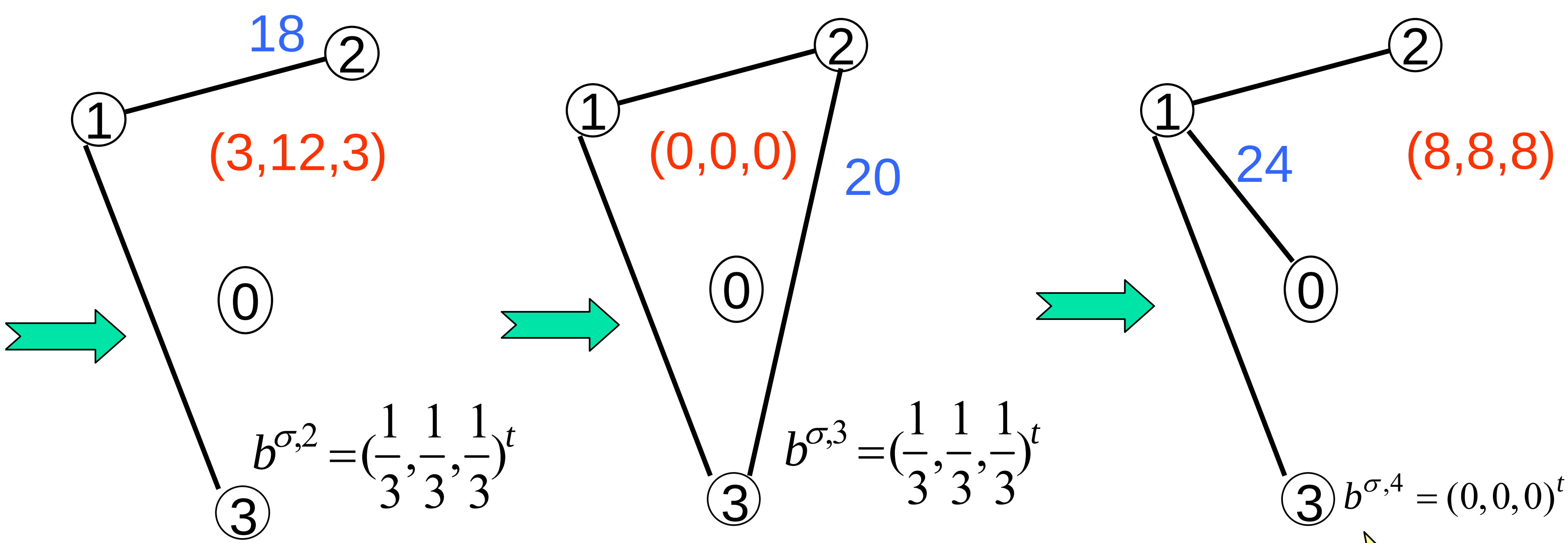


There are no edge costs to share.

$$b^{\sigma,1} = \left(\frac{1}{2},1,\frac{1}{2}\right)^t$$



1 and 3 share cost 10 equally.



2 is connected to 1 and 3 who were already connected: 2 pays 2/3 of 18 whereas the remaining is shared equally between 1 and 3.

Oops... there is a cycle: nobody want it.

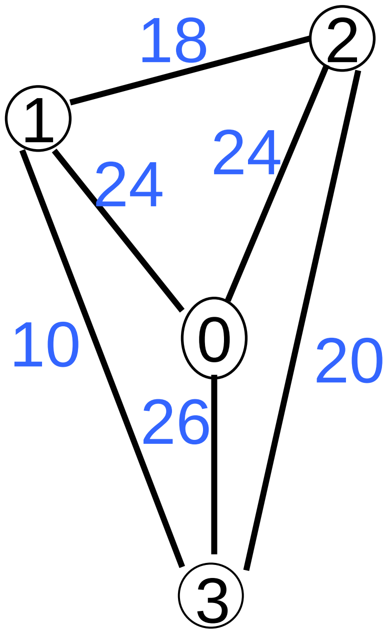
Players are connected to 0: share the total cost of the last edge (=24) equally

Algorithm to calculate the P-value

IDEA: charge the cost of an edge constructed during the Kruskal algorithm only between agents involved, keeping into account the cardinality of the connected components at that step and at the previous step of the algorithm

- At any step of the Kruskal algorithm where a component is connected to some agents, charge the cost of that edge among these agents in the following way:
  - Proportionally to the $\text{cardinality\_current\_step}^{-1}$ if an agent is connected to a component which is connected to the source,
  - Otherwise, charge it proportionally to the difference: $\text{cardinality\_previous\_step}^{-1} - \text{cardinality\_current\_step}^{-1}$

P-value

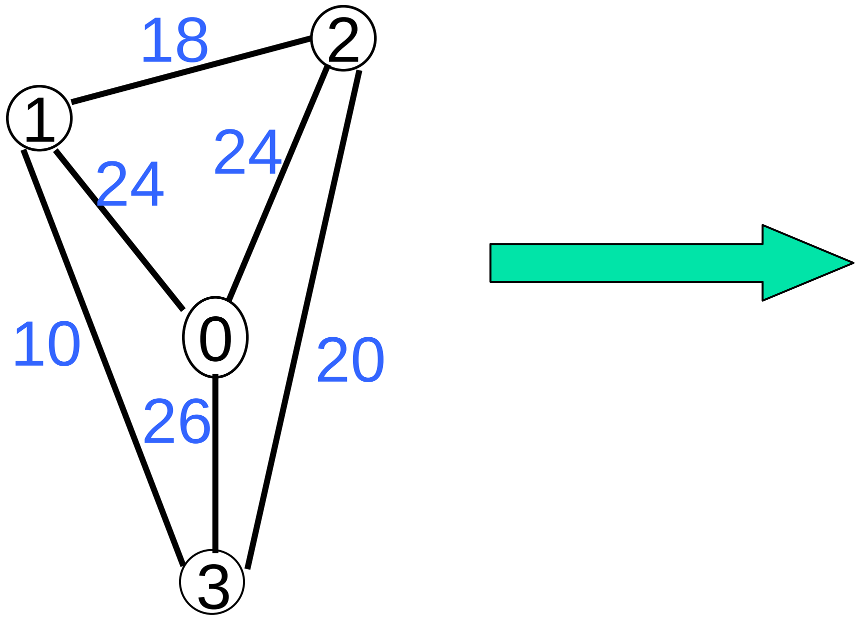


Make the sum of all edge-by-edge allocations:

(0, 0, 0) +
(5, 0, 5) +
(3,12,3) +
(0, 0, 0) +
(8, 8, 8) =

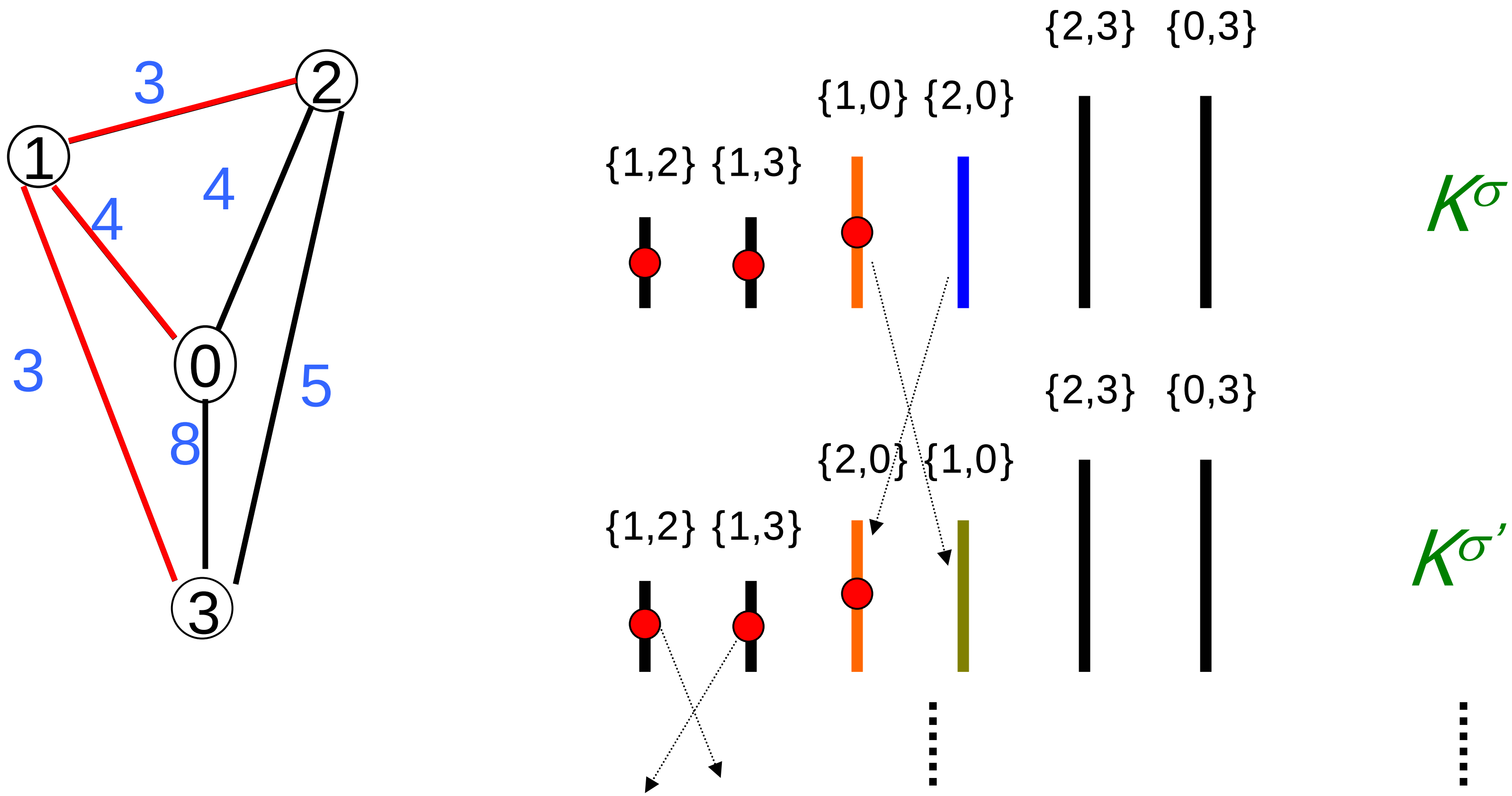
P-value = (16,20,16)

P-value is a function defined on the set of all mcst situations



P=(16, 20,16)

P-value belongs to the core of the corresponding mcst game



3
2
1
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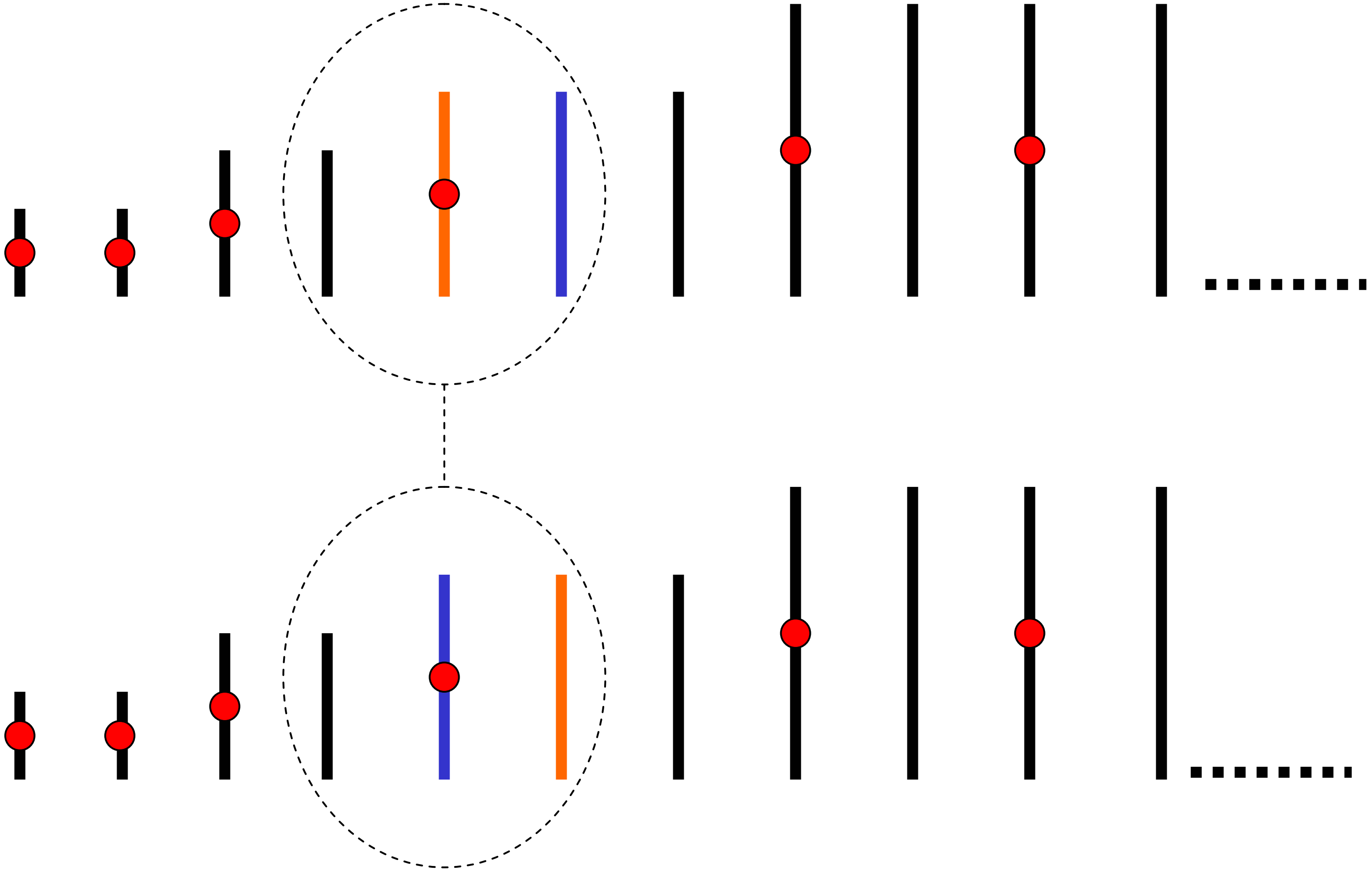
4
3
0
5
8
3
{2,3} {0,3}
{1,0} {2,0}
{1,2} {1,3}
K^{σ}
{2,3} {0,3}
{2,0} {1,0}
{1,2} {1,3}
$K^{\sigma'}$

Proposition 1. *If* $w \in K^{\sigma} \cap K^{\sigma'}$ *with* $\sigma, \sigma' \in \Sigma_{E_{N'}}$ *, then* $P^{\sigma}(w) = P^{\sigma'}(w)$.



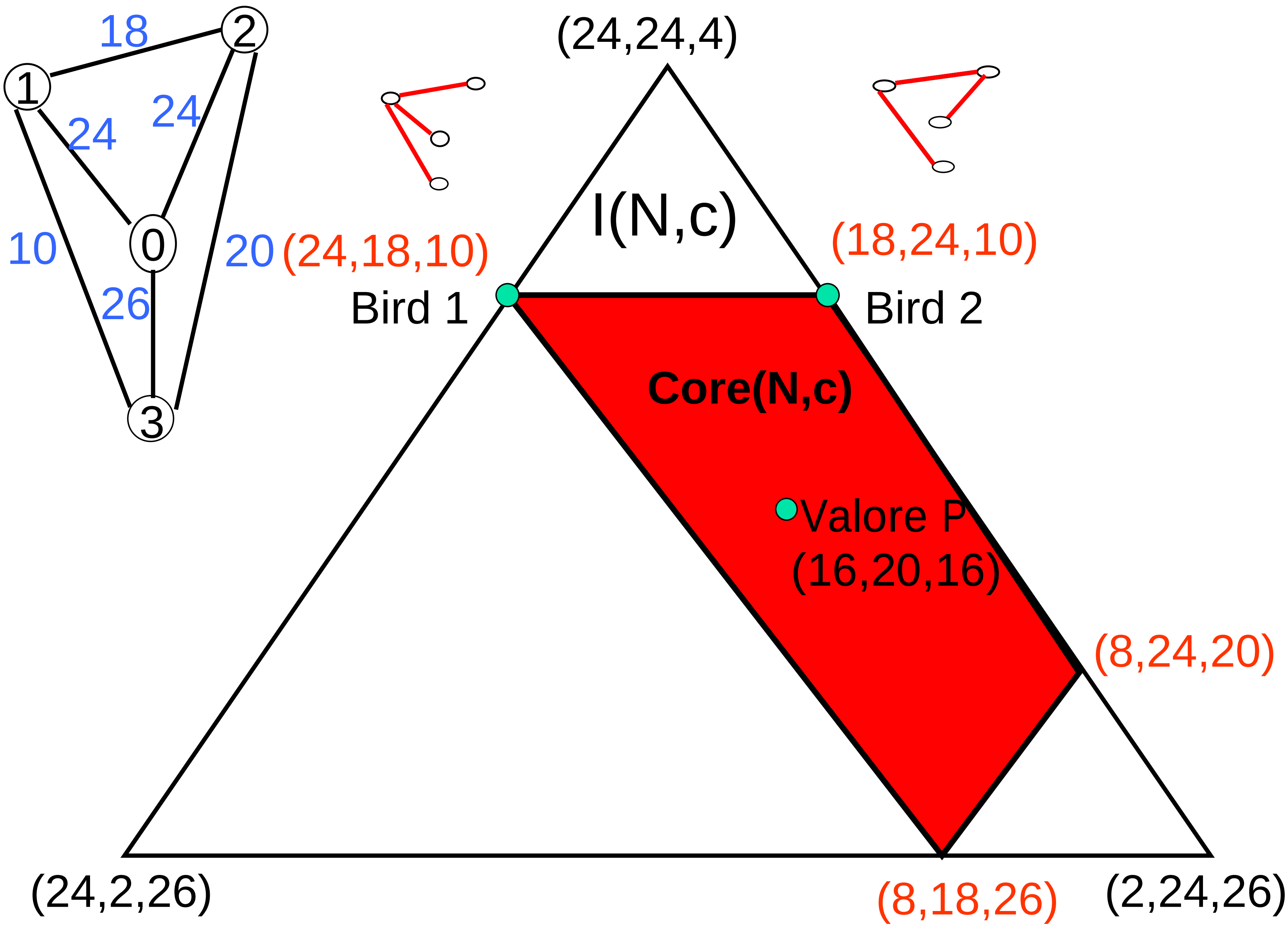
Definition 3. The P-value is the map $P : \mathcal{W}^{N'} \to \Re^{N}$, defined by

$$P(w) = P^{\sigma}(w) = M^{\sigma} w^{\sigma}$$

for each $w \in \mathcal{W}^{N'}$ and $\sigma \in \Sigma_{E_{N'}}$ such that $w \in K^{\sigma}$.

P-value

- Always provides a unique allocation (given a mcst situation).
- It is in the core of the corresponding mcst game.
- Satisfies cost monotonicity.
- Satisfies population monotonicity.



18
24
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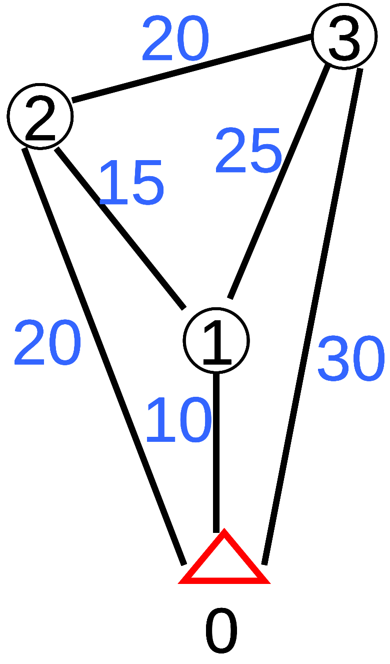
(24,24,4)
I(N,c)
(24,18,10)
(18,24,10)
Bird 1
Bird 2
Core(N,c)
Valore P
(16,20,16)
(8,24,20)
(24,2,26)
(8,18,26)
(2,24,26)

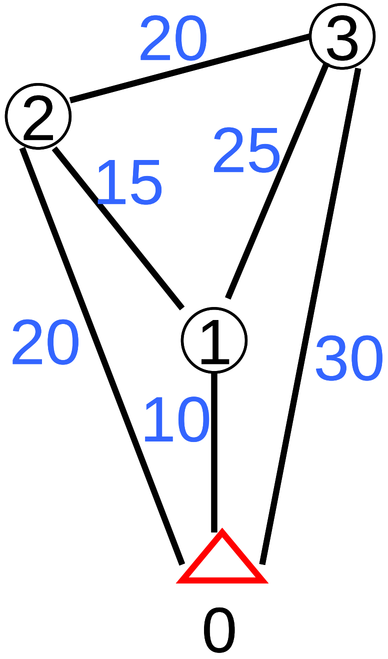
Population Monotonic Allocation Scheme (PMAS)



	1	2	3	
{1,2,3}	10	15	20	c({1,2,3})
{1,2}	10	15	*	c({1,2})
{1,3}	10	*	25	c({1,3})
{2,3}	*	20	20	c({2,3})
{1}	10	*	*	c({1})
{2}	*	20	*	c({2})
{3}	*	*	30	c({3})

$a_{S,i} \geq a_{T,i}$ for all $S,T \in 2^N$ and $i \in N$ with $i \in S \subseteq T$

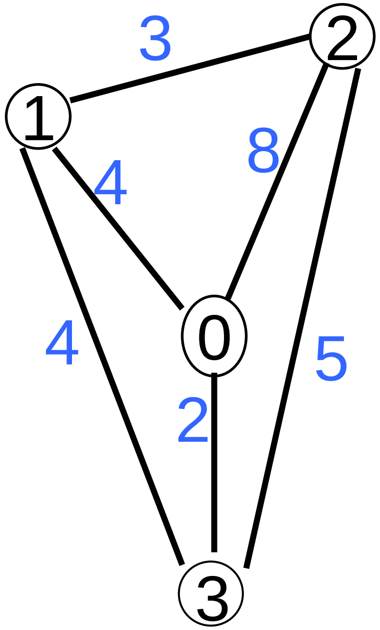
Population Monotonic Allocation Scheme (PMAS)



	1	2	3
{1,2,3}	10	15	20
$\supseteq$	$\leq$	15	$\leq$
{1,3}	10	*	25
{2,3}	*	20	20
{1}	10	*	*
{2}	*	20	*
{3}	*	*	30

$a_{S,i} \geq a_{T,i}$ for all $S,T \in 2^N$ and $i \in N$ with $i \in S \subseteq T$

Exercise:



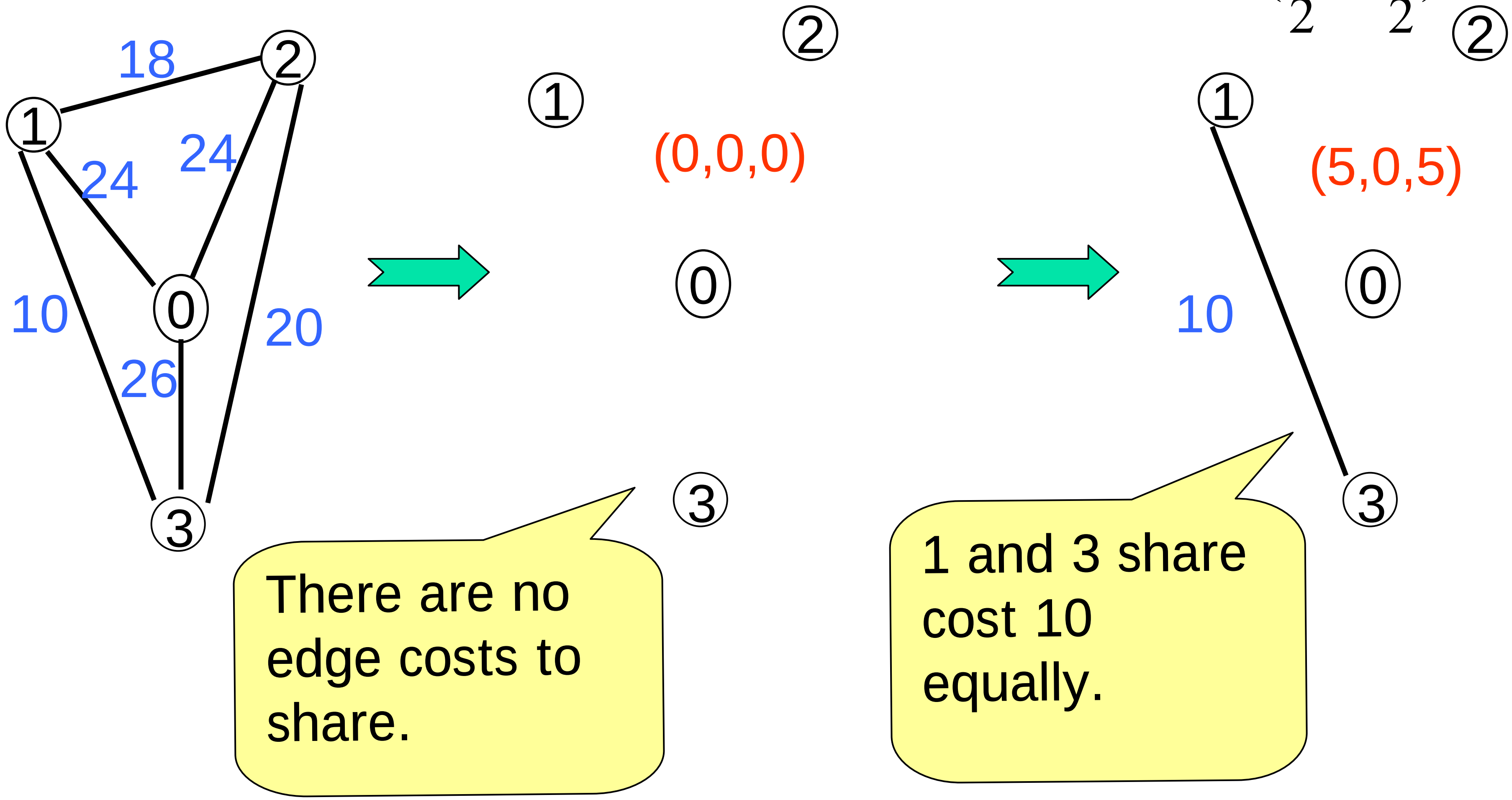
Consider the mcst situation depicted here. Determine:

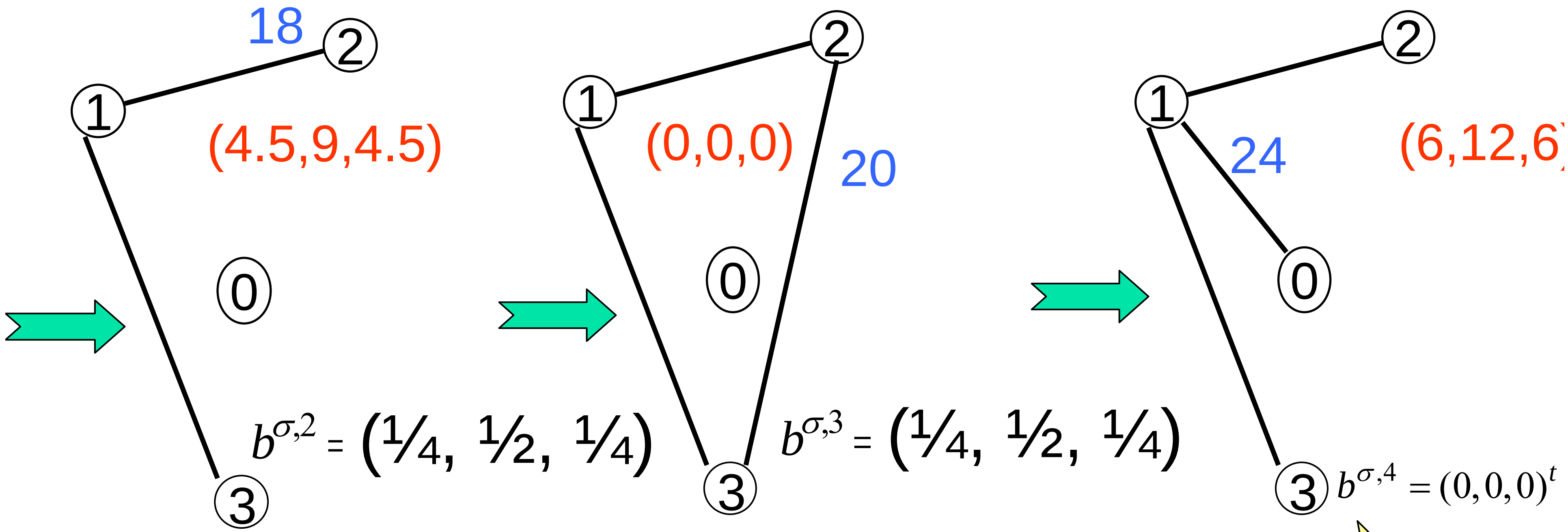
- the corresponding mcst game;
- the core of the mcst game;
- the allocation given by the Bird rule;
- The P-value.

Proportional rule: (Feltkamp (1994))

$$b^{\sigma,0} = (1,1,1)^t$$

$$b^{\sigma,1} = (\frac{1}{2},1,\frac{1}{2})^t$$





2 is connected to 1 and 3 who were already connected: 2 pays 1/2 of 18 whereas the remaining is shared equally between 1 and 3.

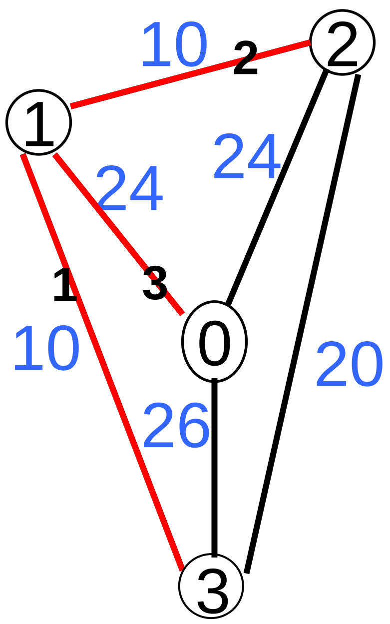
Oops... there is cycle: nobody want it.

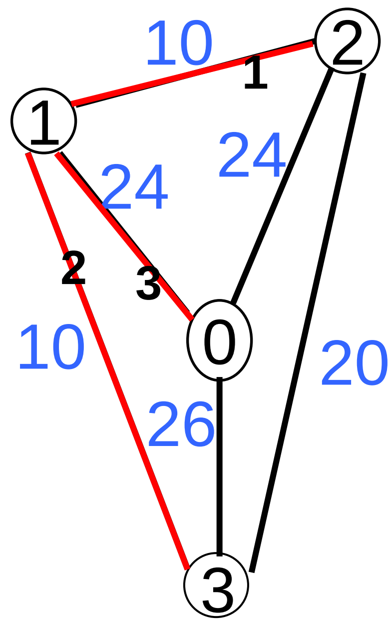
Players are connected to 0 and pay the remaining obligations

The proportional rule is a Construct & Charge rule

Construct & Charge rules are based on the following general cost allocation protocol:

- As soon as a link is constructed in the Kruskal algorithm procedure:
  1) it must be totally charged among agents which are not yet connected with the source (*connection property*)
  2) Only agents that are on some path containing the new edge may be charged (*involvement property*)
- when the construction of a mcst is completed, each agent has been charged for a total amount of fractions equal to 1 (*total aggregation property*).

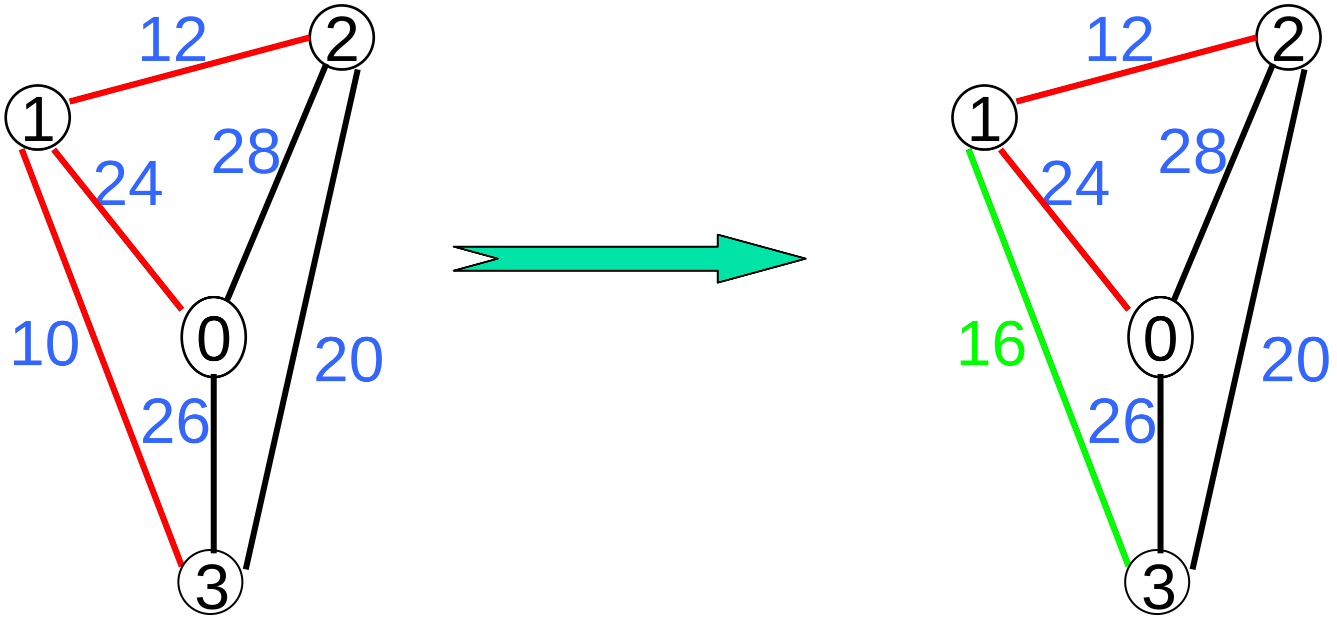




Allocation provided by the proportional rule according to this ordering is (13.5, 17 ,13.5)	Allocation provided by the proportional rule according to this ordering is(13.5, 13.5, 17)

Both allocations are in the core of the corresponding mcst game.

Cost monotonicity: Proportional rule



Proportional rule: (14, 18 ,14) Proportional rule: (16, 16 ,20)

The Proportional rule is not cost monotonic.

Axiomatic characterization (4 independent axioms)

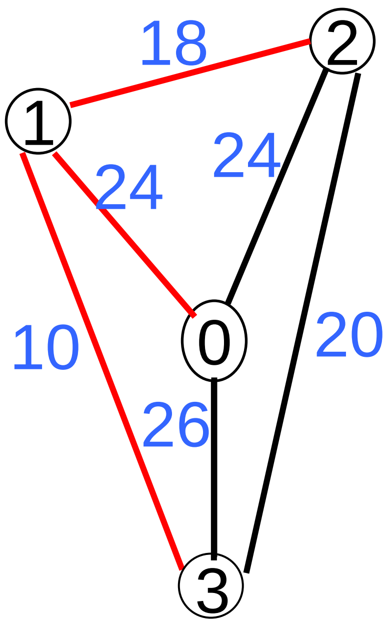
A solution for mcst situations $F : \mathcal{W}^{N'} \rightarrow \Re^N$

Property 1. The solution F is *efficient* (EFF) if for each $w \in \mathcal{W}^{N'}$

$$\sum_{i \in N} F_i(w) = w(\Gamma),$$

where Γ is a minimum cost spanning network on N'.

Example:



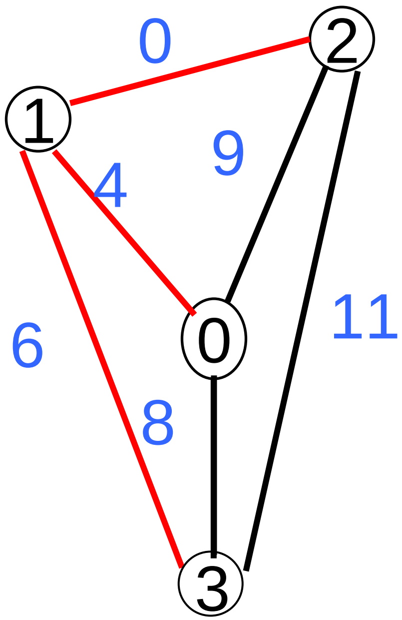
$w(\Gamma)=52$

$$P(w) = M^{\sigma} w^{\sigma} = (16, 20, 16)^t$$

Property 2. The solution F has the *Equal Treatment* (ET) property if for each $w \in \mathcal{W}^{N'}$ and for each $i,j \in N$ with $C_i(w) = C_j(w)$

$$F_i(w) = F_j(w).$$

Example:

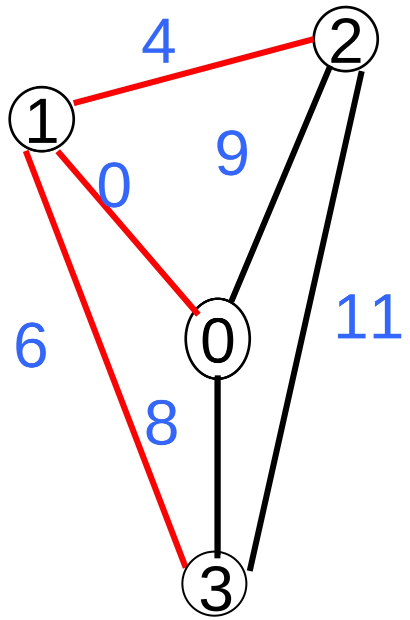


$P(w)=(2,2,6)^t$

Property 3. The solution F has the *Upper Bounded Contribution* (UBC) property if for each $w \in \mathcal{W}^{N'}$ and every (w,N')-component $C \neq \{0\}$

$$\sum_{i \in C \setminus \{0\}} F_i(w) \leq \min_{i \in C \setminus \{0\}} w(\{i,0\}).$$

Example:



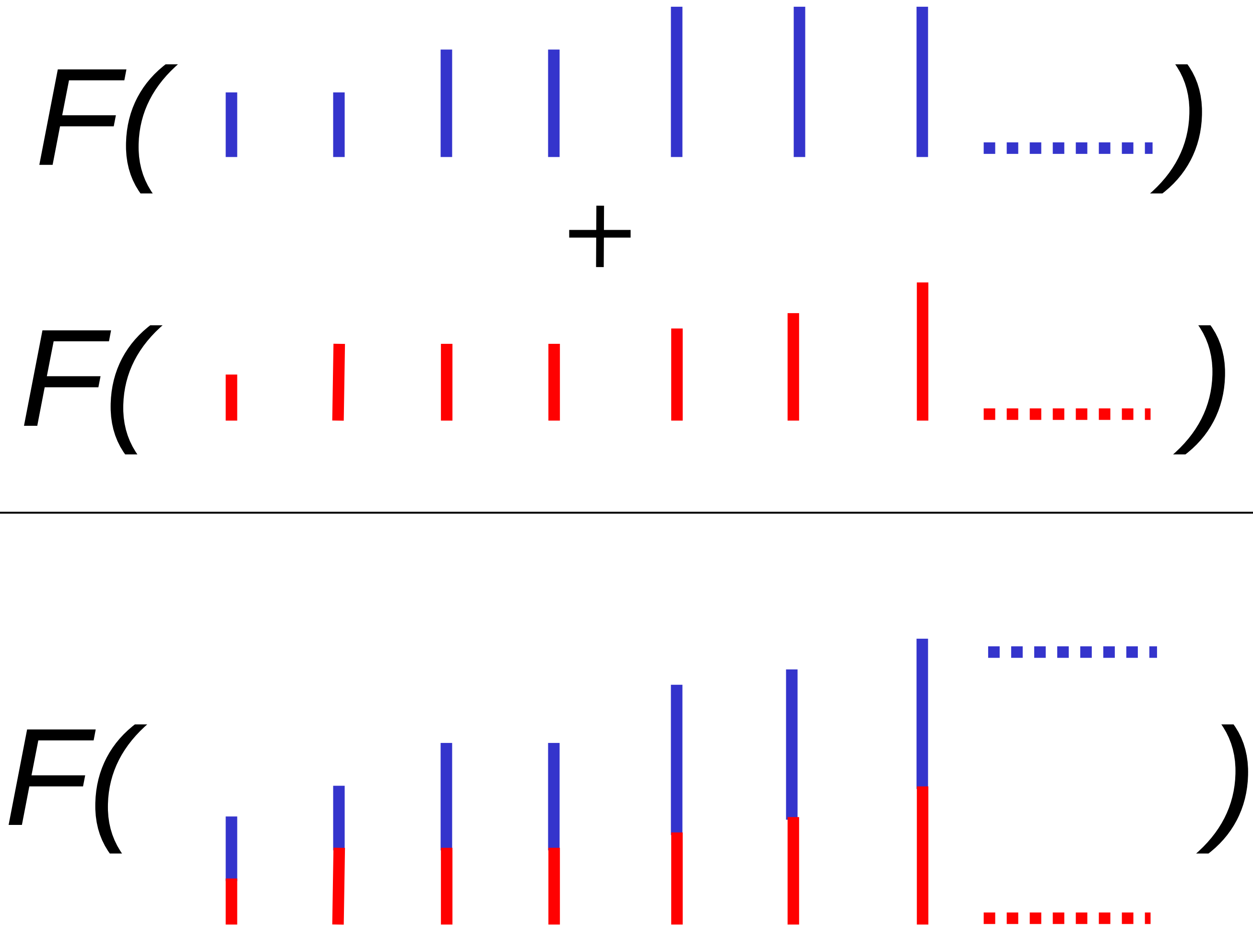
$P(w)=(0,4,6)^t$

Note that 1 is dummy in the corresponding mcst game

Property 4. The solution F has the *Cone-wise Positive Linearity* (CPL) property if for each $\sigma \in \Sigma_{E_{N'}}$, for each pair of mcst situations $w, \hat{w} \in K^{\sigma}$ and for each pair $\alpha, \hat{\alpha} \geq 0$, we have

$$F(\alpha w + \hat{\alpha}\hat{w}) = \alpha F(w) + \hat{\alpha} F(\hat{w}).$$

Example:



Theorem 1. *The P-value is the unique solution which satisfies the properties EFF, ET, UBC and CPL on the class $\mathcal{W}^{N'}$ of mcst situations.*

- It is possible to prove that the P-value satisfies the four properties EFF, ET, UBC and CPL.

- To prove the uniqueness consider a solution for mcst situation F which satisfies EFF, ET, UBC and CPL:

  - first look at the **simple mcst situations** (0-1 cost of edges): on such simple situation, EFF, ET and UBC imply F=P-value;

  - it is possible to decompose each mcst situation as a linear combination of simple mcst problems;

  - by CPL it follows that the F=P-value on each mcst situation.

Phd Thesis, Tilburg Univeristy, The Netherlands:

http://arno.uvt.nl/show.cgi?fid=80868

Shapley value in practice

- In practice, the main difficulty is the effort required for collecting the 2^n data needed to have a TU-game on a set of n players.
- It can happen that the data have a *potentially simple structure*, so that it is possible to treat (in applications) games with a huge number of players.
- in such cases one can exploit the *specific structure* of the data to get a much more manageable formula of the Shapley value

Airport games (Littlechild and Thompson (1977), and Littlechild and Owen (1973))

- The issue is: how to divide the costs due to the landing strip of an airport among the planes that use it?
- One idea has to do with the identification of the players that will give rise to a cooperative (cost) TU-game
- a reasonable modeling approach brings to the idea that the players are the *landings that occur* during the lifetime of the landing strip (or during one year...).

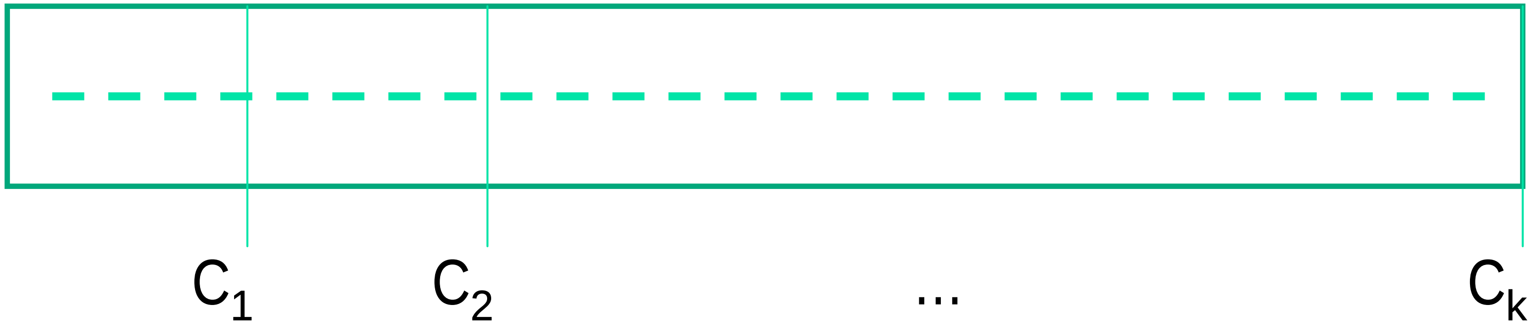
Define the characteristic function

- Since not all players will need a landing strip of the same length, one can reasonably assume that the cost associated with a landing strip long enough to accommodate all of the landings in S can be imputed to S.
- Formally, we partition the set of all landings, N, into groups of landings that require a strip of the same length: $N_1, N_2, \ldots, N_k$, ordered in an increasing way w.r.t. costs.
- For each group N_i, let C_i be its cost. So

$$c(S) = \max\{C_t : S \cap N_t \neq \emptyset\}.$$

How to share airport costs

- Assume there is no worry about the intensity of use of the various components by the players (landings),
- a sensible accounting principle suggests to divide the cost due to an element evenly among those who use it.
- In such a way, it is easy to get a sensible cost allocation, for whose straightforward computation we need very few data:
  - the cardinality of each of the homogeneous groups N_i,
  - $C_1, C_2, \ldots, C_k$ the costs induced by each of the groups.



- The resulting allocation, for a player m belonging to N_i, is given by the formula.

$$\phi_m(v) = \sum_{j=1}^{i} \frac{C_j}{\bigcup_{r=j}^{k} N_r}$$

- an accounting principle, without making any explicit reference to the (cost) game
- this approach provides exactly the Shapley value for the given game

Railway network

- A slight generalization of this line of thought has been provided by Fragnelli et al. (1999) (see also González and Herrero (2004)).
- The issue was, again, a fair imputation of the costs arising from the use of an infrastructure (the railway network, in this case) among its users.
- Here, again, one faces a problem whose modeling leads “naturally” to a game with a lot of players, that in this case are *trains running on the infrastructure* during, e.g., one year.

Many cost elements (1)

- m different services (or cost elements) $E_1, E_2, \ldots, E_m$ (roads, electrical lines, terminals... Try to imagine also internet services...)
- Every player may use one or more services: $P_1, P_2, \ldots, P_m$ cost of the services and S_i the set of players using E_i.
- Consider the game (N,c) where the cost of a non-empty coalition $S \in 2^N$ is the sum of the costs of services used by at least one player:

$$c(S) = \sum_{\substack{1 \le k \le m \\ k : S \cap S_k \neq \emptyset}} P_k$$

Many cost elements (2)

- Classical examples are application for the computation of *service tariffs*: telephone, building, waste or water treatment, etc.
- Game (N,c) can be seen as a sum of m simple cost games $c^1+c^2+ \ldots +c^m$ where c^k is such that:

$$c^k(S) = \begin{cases} P_k \text{ if } S \cap S_k \neq \varnothing \\ 0 \text{ otherwise} \end{cases}$$

- and the Shapley value for such games is

$$\phi_j(c^k) = \begin{cases} \dfrac{P_k}{s_k} \text{ if } j \in S_k \\ 0 \text{ otherwise} \end{cases}$$

Many cost elements (3)

- Consequently, by the additivity of the Shapley value

$$\phi_i(c) = \phi_i(c^1 + c^2 + \ldots + c^k) = \sum_{k:i \in S} \frac{P_k}{s_k}$$

- since c^k are concave games, their sum is a concave game: the Shapley value $\phi(c)$ is in the core of game (N,c) (and also pmas extendable)

Tariffs in practice

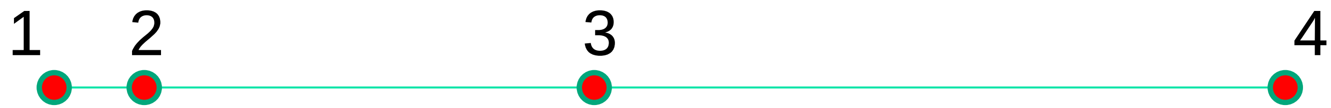
- The use of an allocation method, like the Shapley value, among different municipalities (for example) can provide difficulties when the citizens are called to pay for the services provided.
- It is quite possible that the allocation of costs among municipalities brings to differences in the tariffs for the final users that may be difficult to justify (at least, from the point of view of gathering enough political consensus at the local elections).
- This tension is clearly visible from the interesting interviews of the local decision makers that are mentioned in Loehman et al. (1979).

Water related issues

- The modeling tool of TU-games has been often applied to the context of issues related with water: allocation of water, allocation of costs related with various kinds of projects (water reservoirs, irrigation systems, wastewater treatments and reuse, etc.).
- One of the most relevant contributions, from this point of view, can be traced back to the work of Loehman and Whinston (1976) and related papers (e.g., Loehman et al. 1979).

Against symmetry in applications

- They take into account the fact that, for some structural reason, it is not plausible that all coalitions could be potentially considered. For example, to build a piping system



- it is quite possible that coalitions like $\{1, 4\}$ or $\{2, 4\}$ will not form, possibly due to high costs due to the distance from a "source" located close to 1 and 2, while the presence of player 3 could allow some savings for coalitions like $\{1, 3, 4\}$ and $\{2, 3, 4\}$.

Generalized Shapley value

- taking into account how these asymmetries influence the coalition building process.
- some of the permutations should not be taken into account, since in the process they would require to build up an "impossible" coalition.
- if we assume that coalitions {1, 4} and {2, 4} cannot form, then one should delete the permutations that involve in the process,

1234	2134	3124	~~4123~~
1234	2143	3142	~~4132~~
1324	2314	3214	~~4213~~
1342	2341	3241	~~4231~~
~~1423~~	~~2413~~	3412	4312
~~1432~~	~~2431~~	3421	4321

Players heterogeneity (1)

- Sometimes we need to mix players that are significantly different (e.g., towns and “big” farms).
- This heterogeneity of the players makes the use of the symmetry axiom questionable.
- One reason for asymmetry could be, for example, a different exposure to risk between “players” of different kinds.

Players heterogeneity (2)

- Example: a much higher exposure to risk of farmers, compared with a town, concerning the profits (or savings) obtainable from the facility for the treatment of wastewater.
- An answer to this issue could be to use a more sophisticated model than a classical TU-game, like a *stochastic TU-game* (Suijs and Borm 1999)
- the extension of a solution for TU-games to this richer model is not obvious or unique;
- Timmer et al. (2004): three different definitions of Shapley value applied to stochastic TU-game.