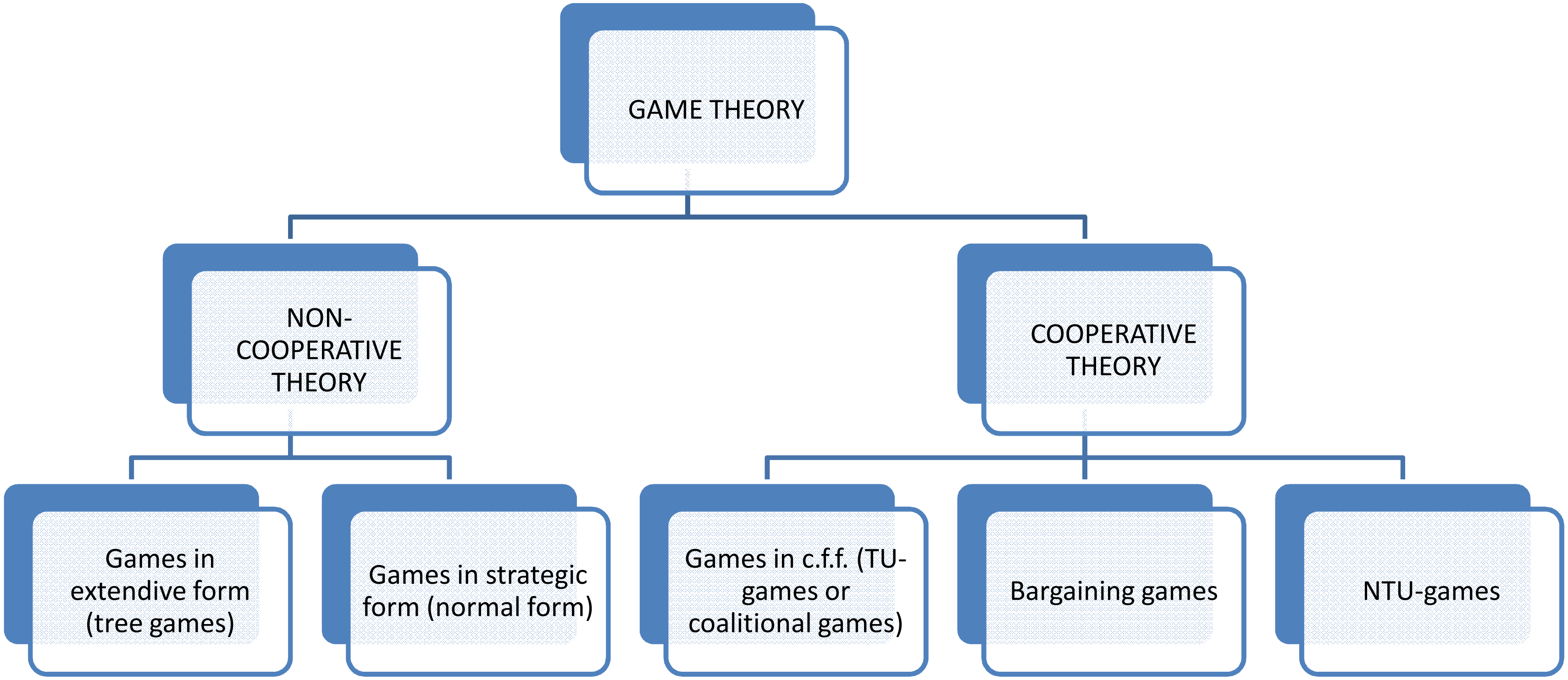
Introduction to Game Theory and Applications

Stefano MORETTI and Fioravante PATRONE
LAMSADE (CNRS), Paris Dauphine
and
DIPTEM, University of Genoa

Paris, Telecom ParisTech, 2010



- Dominant strategies
- Nash eq. (NE)
- Subgame perfect NE
- NE & refinements

…

- Core
- Shapley value
- Nucleolus
- τ-value
- PMAS

….

- Nash sol.
- Kalai-Smorodinsky

….

- CORE
- NTU-value
- Compromise value

…

No binding agreements
No side payments
Q: Optimal behaviour in conflict situations

binding agreements
side payments are possible (sometimes)
Q: Reasonable (cost, reward)-sharing

How to share v(N)…

- The Core of a game can be used to exclude those allocations which are *not stable*.
- But the core of a game can be a bit "*extreme*" (see for instance the glove game)
- Sometimes the core is *empty* (see for example the game with pirates)
- And if it is not empty, there can be many allocations in the core (*which is the best*?)

An axiomatic approach (Shapley (1953)

- Similar to the approach of Nash in bargaining: *which properties an allocation method should satisfy in order to divide v(N) in a reasonable way*?
- Given a subset **C** of $\mathbf{G}^N$ (class of all TU-games with N as the set of players) a *(point map) solution* on **C** is a map Φ:**C** $\rightarrow \mathrm{IR}^N$.
- For a solution Φ we shall be interested in various properties…

Symmetry

PROPERTY 1(SYM) For all games $v \in \mathbf{G}^N$,

If $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in 2^N$ s.t. $i, j \in N \setminus S$,

then $\Phi_i(v) = \Phi_j(v)$.

EXAMPLE

Consider a TU-game $(\{1,2,3\},v)$ s.t. $v(1) = v(2) = v(3) = 0$, $v(1, 2) = v(1, 3) = 4$, $v(2, 3) = 6$, $v(1, 2, 3) = 20$.

Players 2 and 3 are symmetric. In fact:

$v(\varnothing \cup \{2\}) = v(\varnothing \cup \{3\}) = 0$ and $v(\{1\} \cup \{2\}) = v(\{1\} \cup \{3\}) = 4$

If Φ satisfies SYM, then $\Phi_2(v) = \Phi_3(v)$

Efficiency

PROPERTY 2 (EFF) For all games $v \in \mathbf{G}^N$,

$\sum_{i \in N} \Phi_i(v) = v(N)$, i.e., $\Phi(v)$ is a *pre-imputation*.

Null Player Property

DEF. Given a game $v \in \mathbf{G}^N$, a player $i \in N$ s.t.

$v(S \cup i) = v(S)$ for all $S \in 2^N$ will be said to be a null player.

PROPERTY 3 (NPP) For all games $v \in \mathbf{G}^N$, $\Phi_i(v) = 0$ if i is a null player.

EXAMPLE Consider a TU-game $(\{1,2,3\},v)$ such that

$v(1) = 0$, $v(2) = v(3) = 2$, $v(1, 2) = v(1, 3) = 2$, $v(2, 3) = 6$,

$v(1, 2, 3) = 6$. Player 1 is a null player. Then $\Phi_1(v) = 0$

EXAMPLE Consider a TU-game ({1,2,3},v) such that v(1) =0, v(2) = v(3) = 2, v(1, 2) = v(1, 3) = 2, v(2, 3) = 6, v(1, 2, 3) = 6. On this particular example, if Φ satisfies NPP, SYM and EFF we have that

$\Phi_1(v) = 0$ by NPP

$\Phi_2(v)= \Phi_3(v)$ by SYM

$\Phi_1(v)+\Phi_2(v)+\Phi_3(v)=6$ by EFF

So $\Phi=(0,3,3)$

But our goal is to characterize Φ on $\mathbf{G}^N$.

One more property is needed.

Additivity

PROPERTY 2 (ADD) Given $v, w \in \mathbf{G}^N$,

$$\Phi(v)+\Phi(w)=\Phi(v+w).$$

.EXAMPLE Consider two TU-games v and w on N={1,2,3}

v		w		Φ: $v+w$
v(1) =3		w(1) =1		v+w(1) =4
v(2) =4		w(2) =0		v+w(2) =4
v(3) = 1		w(3) = 1		v+w(3) = 2
v(1, 2) =8	+	w(1, 2) =2	=	v+w(1, 2) =10
v(1, 3) = 4		w(1, 3) = 2		v+w(1, 3) = 6
v(2, 3) = 6		w(2, 3) = 3		v+w(2, 3) = 9
v(1, 2, 3) = 10		w(1, 2, 3) = 4		v+w(1, 2, 3) = 14

Theorem 1 (Shapley 1953)

There is a unique point map solution ϕ defined on $\mathbf{G}^N$ that satisfies EFF, SYM, NPP, ADD. Moreover, for any $i \in N$ we have that

$$\phi_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi} m_i^{\sigma}(v)$$

Here Π is the set of all permutations $\sigma: N \to N$ of N, while $m^{\sigma}_i(v)$ is the marginal contribution of player i according to the permutation σ, which is defined as:

$v(\{\sigma(1), \sigma(2), \ldots, \sigma(j)\}) - v(\{\sigma(1), \sigma(2), \ldots, \sigma(j-1)\})$,

where j is the unique element of N s.t. $i = \sigma(j)$.

Probabilistic interpretation: (the "room parable")

- Players gather one by one in a room to create the "grand coalition", and each one who enters gets his marginal contribution.
- Assuming that all the different orders in which they enter are equiprobable, the Shapley value gives to each player her/his expected payoff.

Example

(N,v) such that
N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2)=3,
v(1,3)=1,
v(2,3)=4,
v(1,2,3)=5.

Permutation	1	2	3
1,2,3	**0**	**3**	**2**
1,3,2	**0**	**4**	**1**
2,1,3	**0**	**3**	**2**
2,3,1	**1**	**3**	**1**
3,2,1	**1**	**4**	**0**
3,1,2	**1**	**4**	**0**
Sum	**3**	**21**	**6**
$\phi(v)$	**3/6**	**21/6**	**6/6**

Example

(N,v) such that

N={1,2,3},

v(1)=v(3)=0,

v(2)=3,

v(1,2)=3, v(1,3)=1

v(2,3)=4

v(1,2,3)=5.

Marginal vectors

123→(0,3,2)

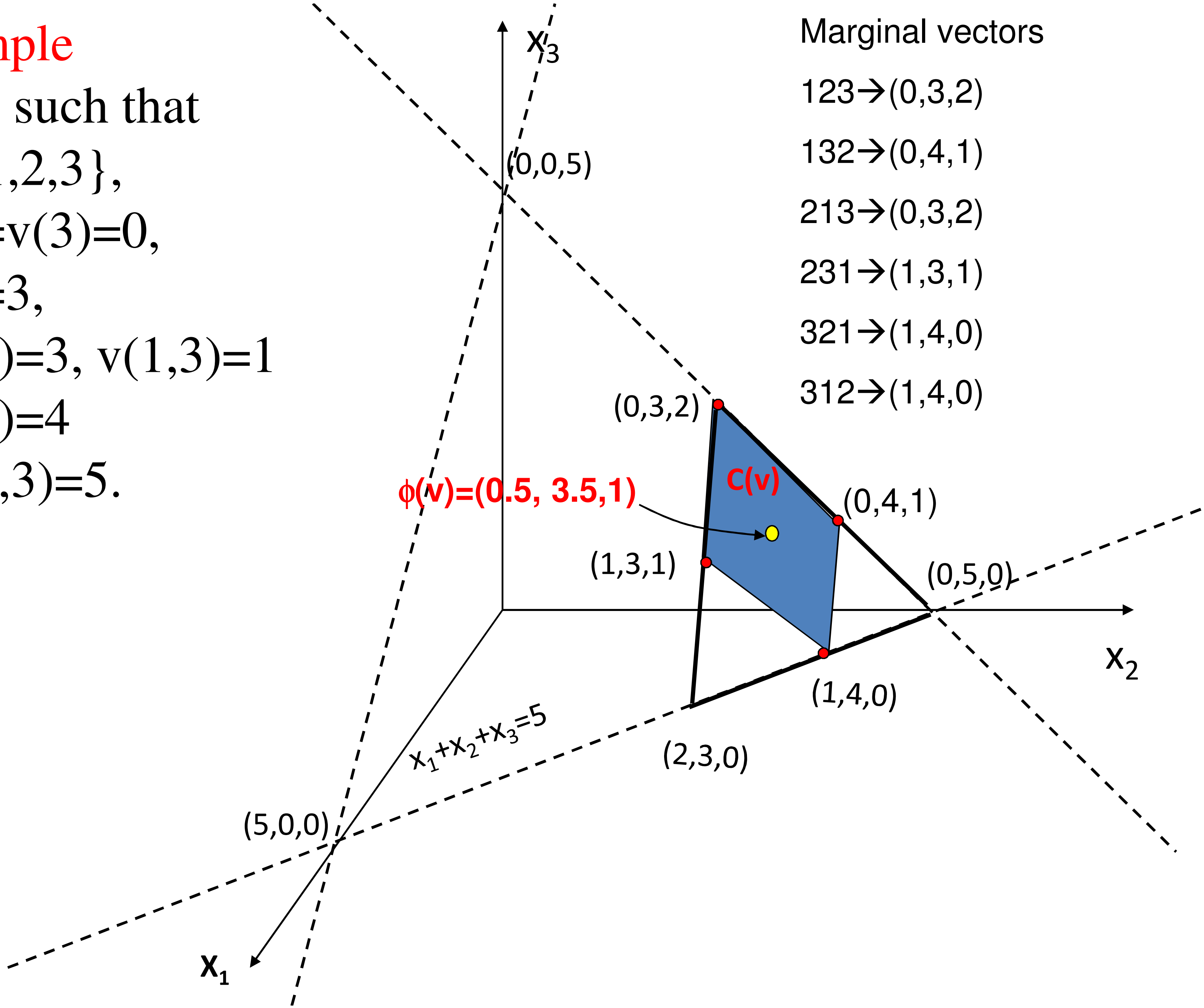
132→(0,4,1)

213→(0,3,2)

231→(1,3,1)

321→(1,4,0)

312→(1,4,0)



Exercise

Calculate the Shapley value of the TU-game (N,v) such that N={1,2,3},

v(1) =3

v(2) =4

v(3) = 1

v(1, 2) =8

v(1, 3) = 4

v(2, 3) = 6

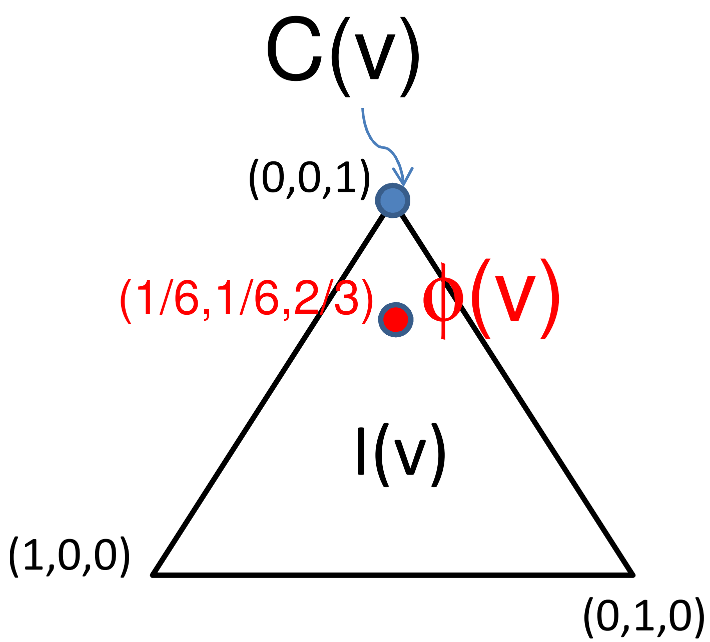
v(1, 2, 3) = 10

Permutation	1	2	3
1,2,3			
1,3,2			
2,1,3			
2,3,1			
3,2,1			
3,1,2			
Sum			
$\phi(v)$			

Example

(Glove game with L={1,2}, R={3})

v(1,3)=v(2,3)=v(1,2,3)=1, v(S)=0 otherwise



Permutation	1	2	3
1,2,3	0	0	1
1,3,2	0	0	1
2,1,3	0	0	1
2,3,1	0	0	1
3,2,1	0	1	0
3,1,2	1	0	0
Sum	1	1	4
$\phi(v)$	1/6	1/6	4/6

Unanimity games (1)

- DEF Let $T\in 2^N\backslash\{\emptyset\}$. The *unanimity game* on T is defined as the TU-game (N,u_T) such that

$$u_T(S)=\begin{cases} 1 \text{ if } T\subseteq S \\ 0 \text{ otherwise} \end{cases}$$

- Note that the class $\mathbf{G}^N$ of all n-person TU-games is a vector space (obvious what we mean for v+w and αv for $v,w\in \mathbf{G}^N$ and $\alpha\in \mathrm{IR}$).
- the dimension of the vector space $\mathbf{G}^N$ is 2^n-1, where $n=|N|$.
- $\{u_T|T\in 2^N\backslash\{\emptyset\}\}$ is an interesting basis for the vector space $\mathbf{G}^N$.

Unanimity games (2)

- Every coalitional game (N, v) can be written as a linear combination of unanimity games in a unique way, i.e., $v = \sum_{S\in 2^N} \lambda_S(v) u_S$.
- The coefficients $\lambda_S(v)$, for each $S\in 2^N$, are called unanimity coefficients of the game (N, v) and are given by the formula: $\lambda_S(v) = \sum_{T\in 2^s} (-1)^{s-t}\, v(T)$.

.EXAMPLE Unanimity coefficients of ({1,2,3},v)

$\lambda_S(v) = \sum_{T\in 2^S} (-1)^{s-t}\, v(T)$

$v(1) = 3$	$\lambda_1(v) = 3$
$v(2) = 4$	$\lambda_2(v) = 4$
$v(3) = 1$	$\lambda_3(v) = 1$
$v(1, 2) = 8$	$\lambda_{\{1,2\}}(v) = -3-4+8=1$
$v(1, 3) = 4$	$\lambda_{\{1,3\}}(v) = -3-1+4=0$
$v(2, 3) = 6$	$\lambda_{\{2,3\}}(v) = -4-1+6=1$
$v(1, 2, 3) = 10$	$\lambda_{\{1,2,3\}}(v) = -3-4-1+8+4+6-10=0$

$$v=3u_{\{1\}}(v)+4u_{\{2\}}(v)+u_{\{3\}}(v)+u_{\{1,2\}}(v)+u_{\{2,3\}}(v)$$

Sketch of the Proof of Theorem1

- Shapley value satisfies EFF, SYM, NPP, ADD ("easy" to prove).
- Properties EFF, SYM, NPP determine ϕ on the class of all games αv, with v a unanimity game and $\alpha \in \mathbb{R}$.
  - Let $S \in 2^N$. The Shapley value of the unanimity game (N,u_S) is given by

$$\phi_i(\alpha u_S) = \begin{cases} \alpha/|S| & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

- Since the class of unanimity games is a basis for the vector space, ADD allows to extend ϕ in a unique way to $\mathbf{G}^N$.

An alternative formulation

- Let $m^{\sigma'}_i(v)=v(\{\sigma'(1),\sigma'(2),\ldots,\sigma'(j)\})- v(\{\sigma'(1),\sigma'(2),\ldots, \sigma'(j-1)\})$, where j is the unique element of N s.t. $i = \sigma'(j)$.
- Let $S=\{\sigma'(1), \sigma'(2), \ldots, \sigma'(j)\}$.
- **Q**: How many other orderings $\sigma\in\Pi$ do we have in which $\{\sigma(1), \sigma(2), \ldots, \sigma(j)\}=S$ and $i = \sigma(j)$?
- **A**: they are precisely $(|S|-1)!\times(|N|-|S|)!$
- Where $(|S|-1)!$ Is the number of orderings of $S\setminus\{i\}$ and $(|N|-|S|)!$ Is the number of orderings of $N\setminus S$
- We can rewrite the formula of the Shapley value as the following:

$$\phi_i(v) = \sum_{S\in 2^N : i\in S} \frac{(s-1)!(n-s)!}{n!}\left(v(S)-v(S\setminus\{i\})\right)$$

Alternative properties

- **PROPERTY 5** (Anonymity, ANON) Let $(N, v) \in \mathbf{G}^N$, $\sigma : N \rightarrow N$ be a permutation. Then, $\Phi_{\sigma(i)}(\sigma v) = \Phi_i(v)$ for all $i \in N$.
  - Here σv is the game defined by: $\sigma v(S) = v(\sigma(S))$, for all $S \in 2^N$.

  Interpretation: The meaning of ANON is that whatever a player gets via Φ should depend only on the *structure* of the game v, not on his "name", i.e., the way in which he is labelled.

- DEF (Dummy Player) Given a game (N, v), a player $i \in N$ s.t. $v(S \cup \{i\}) = v(S) + v(i)$ for all $S \in 2^N$ will be said to be a dummy player.

- **PROPERTY 6** (Dummy Player Property, DPP) Let $v \in \mathbf{G}^N$. If $i \in N$ is a dummy player, then $\Phi_i(v) = v(i)$.

NB: often NPP and SYM are replaced by DPP and ANON, respectively.

Characterization on a subclass

- Shapley and Shubik (1954) proposed to use the Shapley value as a power index,
- ADD property does not impose any restriction on a solution map defined on the class of simple games $\mathbf{S}^N$, which is the class of games such that $v(S) \in \{0,1\}$ (often is added the requirement that $v(N) = 1$).
- Therefore, the classical conditions are not enough to characterize the Shapley–Shubik value on $\mathbf{S}^N$.
- We need a condition that resembles ADD and can substitute it to get a characterization of the Shapley–Shubik (Dubey (1975)) index on S^N:

PROPERTY 7 (Transfer, TRNSF) For any $v,w \in S(N)$, it holds:

$$\Phi(v \vee w)+\Phi(v \wedge w) = \Phi(v)+\Phi(w).$$

Here $v \vee w$ is defined as $(v \vee w)(S) = (v(S) \vee w(S)) = \max\{v(S),w(S)\}$, and $v \wedge w$ is defined as $(v \wedge w)(S) = (v(S) \wedge w(S)) = \min\{v(S),w(S)\}$,

.EXAMPLE Two TU-games v and w on N={1,2,3}

Φ		Φ		$\wedge$ Φ		$\vee$ Φ
v(1) =0		w(1) =1		v∧w(1) =0		v∨w(1) =1
v(2) =1		w(2) =0		v∧w(2) =0		v∨w(2) =1
v(3) = 0		w(3) = 0		v∧w(3) = 0		v∨w(3) = 0
v(1, 2) =1	+	w(1, 2) =1	=	v∧w(1, 2) =1	+	v∨w(1, 2) =1
v(1, 3) = 1		w(1, 3) = 0		v∧w(1, 3) = 0		v∨w(1, 3) = 1
v(2, 3) = 0		w(2, 3) = 1		v∧w(2, 3) = 0		v∨w(2, 3) = 1
v(1, 2, 3) = 1		w(1, 2, 3) = 1		v∧w(1, 2, 3) = 1		v∨w(1, 2, 3) = 1

Reformulations

Other axiomatic approaches have been provided for the Shapley value, of which we shall briefly describe those by Young and by Hart and Mas-Colell.

PROPERTY 8 (Marginalism, MARG) A map $\Psi : \mathbf{G}^N \rightarrow IR^N$ satisfies MARG if, given $v,w \in G^N$, for any player $i \in N$ s.t.

$$v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S) \text{ for each } S \in 2^N,$$

the following is true:

$$\Psi_i(v) = \Psi_i(w).$$

Theorem 2 (Young 1988) There is a unique map Ψ defined on G(N) that satisfies EFF, SYM, and MARG. Such a Ψ coincides with the Shapley value.

.EXAMPLE Two TU-games v and w on N={1,2,3}

v	w
v(1) =3	w(1) =2
v(2) =4	w(2) =3
v(3) = 1	w(3) = 1
v(1, 2) =8	w(1, 2) =2
v(1, 3) = 4	w(1, 3) = 3
v(2, 3) = 6	w(2, 3) = 5
v(1, 2, 3) = 10	w(1, 2, 3) = 4

$w(\emptyset\cup\{3\}) - w(\emptyset) = v(\emptyset\cup\{3\}) - v(\emptyset) = 1$

$w(\{1\}\cup\{3\}) - w(\{1\}) = v(\{1\}\cup\{3\}) - v(\{1\}) = 1$

$w(\{2\}\cup\{3\}) - w(\emptyset) = v(\{2\}\cup\{3\}) - v(\emptyset) = 1$

$w(\{1,2\}\cup\{3\}) - w(\{1,2\}) = v(\{1,2\}\cup\{3\}) - v(\{1,2) = 1$

$\Downarrow$

$$\Psi_3(v) = \Psi_3(w).$$

Potential

- A quite different approach was pursued by Hart and Mas-Colell (1987).
- To each game (N, v) one can associate a real number P(N,v) (or, simply, P(v)), its *potential*.
- The "partial derivative" of P is defined as

$$D^i(P)(N, v) = P(N,v) - P(N\setminus\{i\}, v_{|N\setminus\{i\}})$$

Theorem 3 (Hart and Mas-Colell 1987) There is a unique map P, defined on the set of all finite games, that satisfies:

1) $P(\emptyset, v_0) = 0$,

2) $\Sigma_{i\in N} D^iP(N,v) = v(N)$.

Moreover, $D^i(P)(N, v) = \phi_i(v)$. [$\phi(v)$ is the Shapley value of v]

- there are formulas for the calculation of the potential.
- For example, $P(N,v)=\sum_{S\in 2^N} \lambda_S/|S|$ (*Harsanyi dividends*)

Example

$v(1)=3$ $v(2)=4$ $v(3)=1$ $v(1,2)=8$ $v(1,3)=4$ $v(2,3)=6$ $v(1,2,3)=10$	→ $\lambda_1(v)=3$ $\lambda_2(v)=4$ $\lambda_3(v)=1$ $\lambda_{\{1,2\}}(v)=1$ $\lambda_{\{1,3\}}(v)=0$ $\lambda_{\{2,3\}}(v)=1$ $\lambda_{\{1,2,3\}}(v)=0$	→ $P(\{1,2,3\},v)=3+4+1+1/2+1/2=9$ $P(\{1,2\},v_{\|\{1,2\}})=3+4+1/2=15/2$ $P(\{1,3\},v_{\|\{1,3\}})=3+1=4$ $P(\{2,3\},v_{\|\{2,3\}})=4+1+1/2=11/2$

$$\phi_1(v)=P(\{1,2,3\},v)-P(\{2,3\},v_{|\{2,3\}})=9-11/2=7/2$$
$$\phi_2(v)=P(\{1,2,3\},v)-P(\{1,3\},v_{|\{2,3\}})=9-4=5$$
$$\phi_3(v)=P(\{1,2,3\},v)-P(\{1,2\},v_{|\{2,3\}})=9-15/2=3/2$$

Communication networks

Networks → several interpretations:

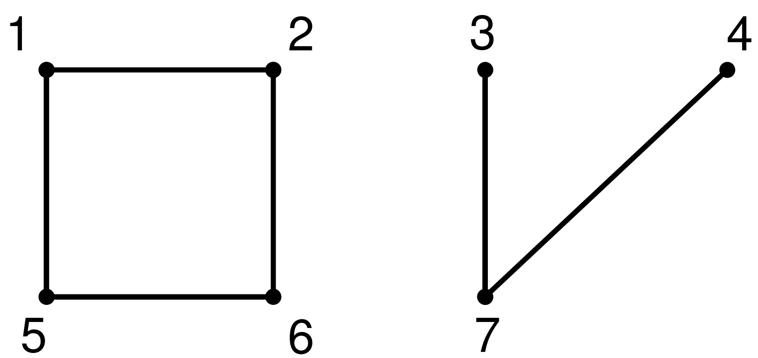
- A way to describe games in extensive form
- Physical connections between individuals, companies, cities…
- Cooperation or communication restrictions between players
  - communications can be described as undirected or directed *graphs*, *hypergraphs*, *partitions.*

Communication networks as undirected graphs:

- An *undirected graph* is a pair (N,L) where
- N is a set of *vertices* (later, *agents* or *players*)
- $L=\{ \{i,j\} \mid \{i,j\}\subseteq N,\ i\neq j \}$ is the set of *edges* (bilateral *communication links*)
- A communication graph (N,L) should be interpreted as a way to model restricted cooperation:
  - Players can cooperate with each other if they are connected (*directly,* or *indirectly* via a path)
  - Indirect communication between two players requires the cooperation of players on a connecting path.

Example

Consider the undirected graph (N,L) with N={1,2,3,4,5,6,7} and L={{1,2}, {2,6}, {5,6}, {1,5}, {3,7}, {4,7}}



Some notations:

L_2 ={{1,2}, {2,6}}

N\L={{1,2,5,6},{3,4,7}}

set of components

L_{-2} ={{5,6}, {1,5}, {3,7}, {4,7}}

N\L_{-2}={{1,5,6},{3,4,7},{2}}

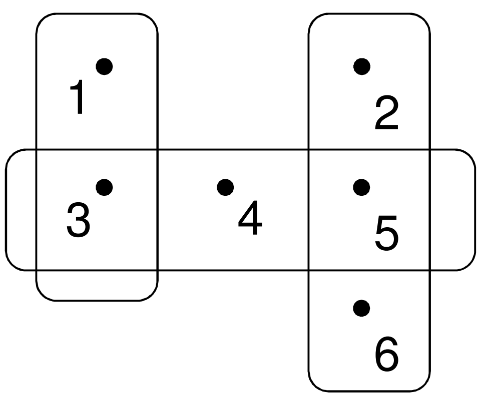
N({{1,2}, {2,6}, {3,7}})={1,2,6,3,7}

Communication within hypergraphs

- A *hypergraph* is a pair (N, C) with N the player set and C a family of subsets of N.
- An element $H \in C$ is called a *conference*.
- Interpretation: communication between players in a hypergraph can only take place within a conference.

Example

Consider the hypergraph (N,C) with N={1,2,3,4,5,6} and C={{1,3}, {3,4,5}, {2,5,6}}



Some notations:

A path from 1 to 2: (1,{1,3},3, {3,4,5},4,{3,4,5},5,{2,5,6},2)

N\C={N} set of components

If R={1,2,3,4,5} then R\C={{1,3,4,5},{2}}

Communication within cooperation structure

- A *cooperation structure* is a pair (N,B) with N the player set and B a partition of the player set N.
- Interpretation: communication between players in a hypergraph can only take place between any subset of an element of the cooperation structure → *Coalition structure (*Aumann and Dréze (1974), Myerson (1980), Owen (1977)).

Cooperative games with restricted communication

- A cooperative game describes a situation in which all players can freely communicate with each other.
- Drop this assumption and assume that communication between players is restricted to a set of communication possibilities between players.
- $L=\{ \{i,j\} \mid \{i,j\}\subseteq N, i\neq j \}$ is the set of *edges* (bilateral *communication links*)
- A communication graph (N,L) should be interpreted as a way to model restricted cooperation:
  - Players can cooperate with each other if they are connected (*directly,* or *indirectly* via a path)
  - Indirect communication between two players requires the cooperation of players on a connecting path.

Communication situations (Myerson (1977))

- A *communication situation* is a triple (N,v,L)
  - (N,v) is a n-person TU-game (represents the economic possibilities of coalitions)
  - (N,L) is a communication network (represents restricted communication possibilities)
- The *graph-restricted game* (N,v^L) is defined as

$$v^L(T)=\sum_{C\in T\backslash L} v(C)$$

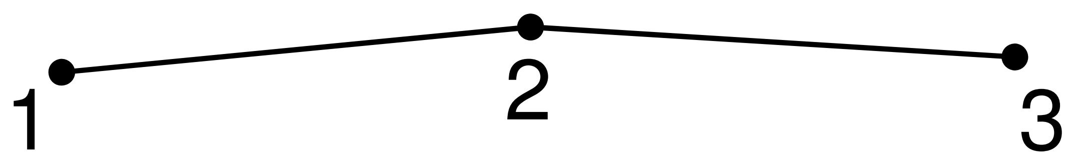
For each $S\in 2^N\backslash\{\varnothing\}$.

Recall that T\L is the set of maximal connected components in the restriction of graph (N,L) to T

Example

A weighted majority game ({1,2,3},v) with the winning quote fixed to 2/3 is considered. The votes of players 1, 2, and 3 are, respectively, 40%, 20%, and 40%. Then, v(1,3)=v(1,2,3)=1 and v(S)=0 for the remaining colitions.

The communication network is



Then,

$v^L(1,2,3)=1$, and $v^L(S)=0$ for the other coalitions.

Solutions for communication situations

- Myerson (1977) was the first to study solutions for communication situations.
- A solution Ψ is a map defined for each communication situation (N,v,L) with value in $IR^{N.}$

PROPERTY 9 Component Efficiency (CE)

For each communication situation (N,v,L) and all $C \in N \backslash L$ it holds that

$$\sum_{i \in S} \Psi_i(N,v,L) = v(C).$$

- Property 9 is an "efficiency" condition that is assumed to hold only for those coalitions whose players are able to communicate effectively among them and *are not connected to other players*. (maximal connected components)

Solutions for communication situations

PROPERTY 10 Fairness (F) For each communication situation (N,v,L) and all $\{i,j\} \in L$ it holds that

$$\Psi_i(N,v,L) - \Psi_i(N,v,L\setminus\{\{i,j\}\}) = \Psi_j(N,v,G) - \Psi_j(N,v,L\setminus\{\{i,j\}\}).$$

- Property 10 says that two players should gain or lose in exactly the same way, when a direct link between them is established (or deleted).

Myerson value

Theorem 4 (Myerson (1977))

There exists a unique solution $\mu(N,v,L)$ which satisfies CE and F on the class of communication situations. Moreover,

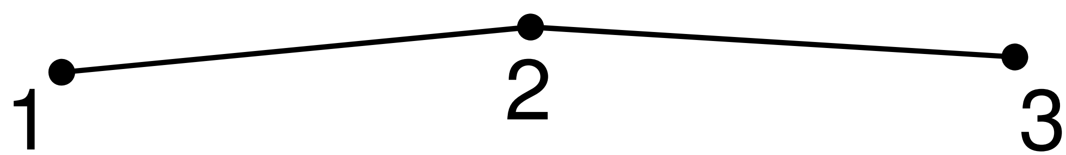
$$\mu(N,v,L)=\phi(v^L)$$

where $\phi(v^L)$ is the shapley vale of the graph-restricted game v^L.

Example

A weighted majority game $(\{1,2,3\},v)$ with the winning quote fixed to 2/3 is considered. The votes of players 1, 2, and 3 are, respectively, 40%, 20%, and 40%. Then, $v(1,3)=v(1,2,3)=1$ and $v(S)=0$ for the remaining colitions.

The communication network is



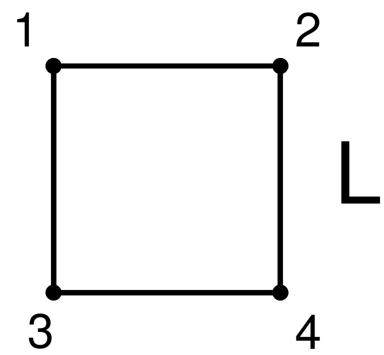
Then,

$v^L(1,2,3)=1$, and $v^L(S)=0$ for the other coalitions.

We have that

$\phi(v)=(1/2,0,1/2)$ and $\mu(N,v,L)= \phi(v^L)=(1/3,1/3,1/3)$.

Example (N,v,L) communication situation such that L is the following network and $v=u_{\{2,4\}}$



Note that, for instance, $v^L(2,4)=v(2)+v(4)=0$.

Easy to note that that $v^L=u_{\{1,2,4\}}+u_{\{2,3,4\}}-u_N$

Therefore,

$\mu(N,v,L)=\phi(vL)=(1/3,2/3,1/3,2/3)-(1/4,1/4,1/4,1/4)$

$=(1/12,5/12,1/12,5/12)$

Application to social networks

- An application of the Shapley value, which uses both the classical one and the one by Myerson (1977), has been proposed by Gómez et al. (2003), to provide a definition of *centrality* in social networks.
- The proposal is to look at the difference between:
  - $\mu(N,v,L)$: the Myerson value, that takes into account the communication structure;
  - $\phi(v)$: the Shapley value, that disregards completely the information provided by the graph *L*.

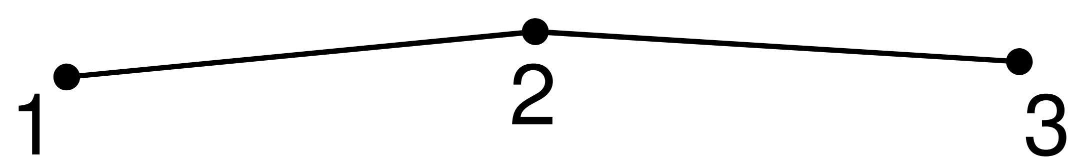
Games and Centrality

- The centrality of a node refers to the *variation* in power due to the social situation (represented by the graph),
- the power is measured using game theory
- More precisely, it is the Shapley value of a game that is used as a power index.
- Gómez et al. (2003) describe general properties of their centrality measure, and in particular, how the abstract structure of the graph influences it.

Example

A weighted majority game $(\{1,2,3\},v)$ with the winning quote fixed to 2/3 is considered. The votes of players 1, 2, and 3 are, respectively, 40%, 20%, and 40%. Then, $v(1,3)=v(1,2,3)=1$ and $v(S)=0$ for the remaining colitions.

The communication network is



We have seen that

$\phi(v)=(1/2,0,1/2)$ and $\mu(N,v,L)=\phi(v^L)=(1/3,1/3,1/3)$.

So, the centrality value is 1/3 for player 2 and –1/6 for 1 and 3.