Introduction to Game Theory and Applications

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Summary

Top secret

The examples will be revealed only in the course.

Relevant characteristics

Decision makers (=players) engaged in an interactive decision problem:
- more than one decision maker (DM) (=player). [The "easy case", 1 DM, is left to Decision Theory (DT)]
- the result is determined by the choices made by each player
- the decision makers' preferences w.r.t. outcomes are (generally speaking) different.

Classical assumptions about players: rational and intelligent.

Relevant “parameters”

- in the three examples, choices are “contemporary”
- players know the relevant data of the interaction decision problem:
- - available strategies
- - payoffs (!)
- - rationality and intelligence of each player
- each player knows that all players know what is listed above
- each player knows that each player knows...

COMMON KNOWLEDGE

- not available the possibility of binding agreements:

NON COOPERATIVE GAMES

Basic model

$(X, E, h, \sqsupseteq)$ where:
- X set of alternatives (choices) available to the DM
- E set of outcomes
- $h : X \to E$ maps alternatives into outcomes

- $\sqsupseteq$ total preorder on E (math object to describe the preferences on E of the DM)

Total preorder on E is a binary relation on E which is:
- reflexive: $\forall x \in E : x \sqsupseteq x$
- transitive: $\forall x, y, z \in E : (x \sqsupseteq y \wedge y \sqsupseteq z) \Rightarrow x \sqsupseteq z$
- total: $\forall x, y \in E : (x \sqsupseteq y \vee y \sqsupseteq x)$

Very important remark: the *rationality* assumption is essentially subsumed in the *transitivity* condition.

Utility functions

Under "reasonable" conditions on E and $\sqsupseteq$ there exists $u : E \to \mathbb{R}$ s.t.:
$\forall x, y \in E : x \sqsupseteq y \Leftrightarrow u(x) \geq u(y)$

We say that the (utility) function u *represents* the preferences of the DM

"Reasonable" conditions are:
- E is countable
- "continuity" conditions on E (topological space) and $\sqsupseteq$

u is not unique. E.g., given *any* strictly increasing $\phi : \mathbb{R} \to \mathbb{R}$, $\phi \circ u$ represents also $\sqsupseteq$

Squeezing the model

We have: $(X, E, h, \sqsupseteq)$.
Using utility functions, we get: (X, E, h, u). To which we can associate the diagram:

$$X \xrightarrow{h} E \xrightarrow{u} \mathbb{R}$$

Composition of functions... $f = u \circ h$:

$$X \xrightarrow{f} \mathbb{R}$$

Or, (X, f). Nice simplification.

Did we dispose of the baby, together with the wastewater?
NO, if we are not interested in dividing objective and subjective part
YES, of paramount importance for *mechanism design*

Relevant characteristics, I: "objective" part

Basic model uses the idea of *state of nature*: $(X, S, E, h, \sqsupseteq)$.
- X as before (alternatives)
- S set of the "states of nature"
- E as before (outcomes)
- $h : X \times S \to E$ maps any couple (alternative, state of nature) to an outcome.

A useful matrix:

$X \backslash S$	s_1	$\dots$	s_n
x_1	e_{11}	$\dots$	e_{1n}
$\dots$	$\dots$	$\dots$	$\dots$
x_m	e_{m1}	$\dots$	e_{mn}

Relevant characteristics, II: "subjective" part

- $\sqsupseteq$ as before? NO! Nonsense! Preferences on E are not enough.

Example: Betting on a dice.

Two alternatives:
- bet 1 euro: if 5 shows, you have a *net* gain of 3 euro, if not, you lose your euro
- no bet: nothing happens

$X \backslash S$	1	2	3	4	5	6
to bet	-1	-1	-1	-1	3	-1
not to bet	0	0	0	0	0	0

Relevant characteristics, II: "subjective" part

Relevant outcomes are: $-1, 0, 3$ (euro). So, we can take $E = \{-1, 0, 3\}$.
Assume the DM is a "standard person", that is:

$$3 \sqsupset 0 \sqsupset -1$$

What can we say? NOTHING relevant for our problem.

Preferences must be *on the rows of the matrix*!
Do you prefer the first row or the second one?

Where are we going to?

Relevant characteristics, II: “subjective” part

Just add another line to the matrix:

	p_1	$\dots$	p_n
$X \backslash S$	s_1	$\dots$	s_n
x_1	e_{11}	$\dots$	e_{1n}
$\dots$	$\dots$	$\dots$	$\dots$
x_m	e_{m1}	$\dots$	e_{mn}

Probability that a state of nature will be the true one.
So, choosing $x \in X$ has the effect of inducing a probability distribution on E.

If we have $u : E \to \mathbb{R}$, to x we can associate:

$$\sum_{i=1}^{n} p(s_i)u(h(x, s_i))$$

the *expected utility* from choosing x.

Our DM has simply to choose $\bar{x} \in X$ that maximizes the expected utility.

The dice example

Let's consider the dice example.

The utility values associated to the two possible choices are (notice that $S = \{1, \dots, 6\}$):

- $x =$ "to bet": $\sum_{i=1}^{6} \frac{1}{6} u(h(x, i)) = \frac{1}{6} u(3) + \frac{5}{6} u(-1)$
- $x =$ "not to bet": $\sum_{i=1}^{6} \frac{1}{6} u(h(x, i)) = \frac{1}{6} 6u(0) = u(0)$

Notice that "to bet" induces a probability distribution ("lottery") on E. This lottery assigns probability $\frac{1}{6}$ to outcome 3, and $\frac{5}{6}$ to outcome -1. Similarly, the lottery induced by "not to bet" assigns probability 1 to 0.

Relevant characteristics, II: “subjective” part

What did we leave out?
- from where probabilities on S do come?

- - are “given”, “objective” probabilities (prob that we get 5 from the dice is 1/6).
von Neumann and Morgenstern (vNM) developed the theory for this case: decision under *risk*

- - are “subjective” (prob that this nice girl will marry me in 2010?)
Savage deduces subjective probabilities from preferences on the rows: decision under *uncertainty*

Relevant characteristics, II: "subjective" part

- utility functions are "vNM" utility functions!
- - are defined up to an affine (strictly increasing) transformation (only $\phi(t) = at + b$. with $a > 0$ is OK).
- - the computation of expected utility is a meaningful operation for them

Dice example ($e \in E = \{-1, 0, 3\}$):
$u(e) = \sqrt{e+1}$; $u(e) = e + 1$; $u(e) = (e+1)^2$.

Risk averse, neutral, loving.

Risk aversion and friends

Assume that $E \subseteq \mathbb{R}$ is an interval.

Remark: Convex subsets of $\mathbb{R}$ are *precisely* the intervals.

Consider the choice between:
- $\bar{e} \in E$ (it is the lottery that assigns probability 1 to $\bar{e}$)
- a "lottery" on E: $(p_1, e_1; \ldots; p_n, e_n)$ such that the expected value of this lottery is precisely $\bar{e}$ ($\sum_{i=1}^n p_i e_i = \bar{e}$)

We can have three cases:
- $\sum_{i=1}^n p_i u(e_i) < u(\bar{e})$: risk adverse DM
- $\sum_{i=1}^n p_i u(e_i) = u(\bar{e})$: risk neutral DM
- $\sum_{i=1}^n p_i u(e_i) > u(\bar{e})$: risk loving DM

Remarks on risk aversion and friends

Remark: of course, the two lotteries that we consider can be induced by the choice of two alternatives in X ("to bet", "not to bet"; buy shares or buy corporate bonds; to buy the rights for drilling in some area, or not).

Remark: the assumption that $E \subseteq \mathbb{R}$ is quite important. It assures us that it is meaningful to make the calculations $\sum_{i=1}^{n} p_i e_i$. A typical case is when elements of E are amounts of money.

Question: in the dice example we had $E \subseteq \mathbb{R}$, but E is not an interval. Why we were speaking of risk neutrality et cetera?

Game form

A game form (in strategic form), with two players, is: (X, Y, E, h).

New aspects w.r.t. decision theory:
- two DMs (we shall call them "players"), so two sets of available alternatives (choices, but here we use the word "strategies")
- $h : X \times Y \to E$ is the map that converts a couple of strategies (one for each player) into an outcome.

Easy to generalize to a finite set of players N: $(N, (X_i)_{i \in N}, E, h)$.
With $h : \prod_{i \in N} X_i \to E$.

Preferences of the players

To get a game we need a second ingredient, the preferences of the players.

If we have two players (called I and II), each will have his preferences.
So: $\sqsupseteq_I, \sqsupseteq_{II}$.
Each one is a total preorder on E.
We shall represent them by utility functions: u and v.

We shall often make the assumption that these utility functions are *vNM*.

Game in strategic form

Patching all together (game form + preferences)...

We use utility functions. In the 2 players case:
(X, Y, E, h, u, v).

The corresponding diagram:

$$\begin{tikzcd}
 & & \mathbb{R} \\
X \times Y \arrow[r, "h"] & E \arrow[ur, "u"] \arrow[dr, "v"'] & \\
 & & \mathbb{R}
\end{tikzcd}$$

Game in strategic form: squeezed

Still in the 2 players case:
(X, Y, f, g).

Where $f = u \circ h$ and $g = v \circ h$.

The squeezed diagram:

$$\begin{tikzcd} & & \mathbb{R} \\ X \times Y \arrow[urr, "f"] \arrow[drr, "g"'] & & \\ & & \mathbb{R} \end{tikzcd}$$

Looking for a solution

What a player will/should do?

- “will”: the descriptive point of view. Aiming at predicting what players will do in the model and hence in the real game
- “should”: the normative point of view. Rationality is based on a *teleological* description of the players. Players have an “end” (not like apples, stones, molecules). So, they could do the “wrong” thing. We could give them suggestions.

Would you suggest a stone what to do? Very relevant difference between hard sciences and social sciences.

Back to the “solution” issue. Can we say something on the basis of our assumptions?

Domination among strategies

From decision theory we borrow the idea of domination among strategies:
- x_1 is (obviously) better than x_2 if:

$h(x_1, s) \sqsupset h(x_2, s)$ **for every** $s \in S$

Trivial modification:
$h(x_1, y) \sqsupset h(x_2, y)$ **for every** $y \in Y$

We shall say that x_1 (strongly) dominates x_2.
So, if x_1 dominates any other $x \in X$, then x_1 **is the solution**

Prisoner's dilemma

The game is:

$I \backslash II$	L	R
T	3 3	1 4
B	4 1	2 2

Obviously B and R are dominant strategies (for I and II respectively).
So, we have the solution. Nice and easy.

But... **the outcome is inefficient!**
Both players prefer the outcome deriving from (T, L).

And so? The problem is that players are (assumed to be) rational and intelligent.

Strategies to avoid

A strategy which is (strongly) dominated by another one will not be played.

So we can delete it. But then could appear *new* (strongly) dominated strategies for the other player. We can delete them. And so on...

Maybe players are left with just one strategy each. Well, a new way to get a solution for the game.

Technically: solution via iterated elimination of dominated strategies.

Strategies to avoid: examples

Example 1:

I \ II	L		R	
T	2	1	1	0
M	1	1	2	0
B	0	0	0	1

Example 2:
Beauty contest.

Important:
- Common knowledge (CK) issues, as seen in the beauty contest.
- Dominance approach: *no conjecture is needed about what the others could do*.

More on dominance

We have three different dominance relations. Given $x_1, x_2 \in X$, we shall say that:
- x_1 strongly dominates x_2 if $f(x_1, y) > f(x_2, y)$ for all $y \in Y$.
- x_1 strictly dominates x_2 if $f(x_1, y) \geq f(x_2, y)$ for all $y \in Y$ and there exists $y \in Y$ s.t. $f(x_1, y) > f(x_2, y)$.
- x_1 weakly dominates x_2 if $f(x_1, y) \geq f(x_2, y)$ for all $y \in Y$.

Iterated elimination of *strongly* dominated strategies is *independent of the order of elimination* (provided that at any round at least one strongly dominated strategy is deleted, if any exists). This is not true for strictly dominated strategies.

Remark: the terminology in use in GT is different from that used here. Usually strongly dominance is referred as "dominance" or "strict dominance", while strict dominance is called "weak dominance", and weak dominance is not mentioned at all.

More on dominance

Exercise: provide an example of "non independence of the order of elimination" for strictly dominated strategies.

Exercise: provide an example of a game with a NE $(\bar{x}, \bar{y})$ s.t. both $\bar{x}$ and $\bar{y}$ are strictly dominated strategies.

Exercise: provide an example in which iterated elimination of strictly dominated strategies eliminates a NE.

Nash equilibrium

Basic solution concept, for games in strategic form.

(2 players only) Given $G = (X, Y, f, g)$, $(\bar{x}, \bar{y}) \in X \times Y$ is a Nash equilibrium for G if:

- $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$
- $g(\bar{x}, \bar{y}) \geq g(\bar{x}, y)$ for all $y \in Y$

Existence: Nash's theorem: mixed strategies... See later
Difficulties:
- not uniqueness
- some Nash equilibria are not sensible (tomorrow)
- inefficiency (Adam Smith was wrong?)

Nash equilibrium and dominance

Theorem:
If a game has a unique couple that survives iterated elimination of dominated strategies, that this couple is a Nash equilibrium.

In particular, a couple of dominating strategies is a Nash equilibrium.

So, in the prisoner's dilemma, (B, R) is the (NB: unique) Nash equilibrium.

So, Nash equilibrium can give an outcome which is inefficient.

Nash equilibrium and dominance

Remark: it happens that "externalities" are relevant!

Interesting decomposition for the PD:

$I \backslash II$	L	R
T	0 0	1 1
B	1 1	2 2

$+$

$I \backslash II$	L	R
T	3 3	0 3
B	3 0	0 0

Note for later (Wednesday): PD is a potential game.

Nash equilibria: examples

Example (battle of the sexes, BoS): $(T, L), (B, R)$ N.E. Not unique, in an essential way

$I \backslash II$	L	R
T	2 1	0 0
B	0 0	1 2

Example (coordination game): $(T, L), (B, R)$ N.E. Not unique, in an essential way

$I \backslash II$	L	R
T	1 1	0 0
B	0 0	1 1

Nash equilibrium: not unique

The battle of the sexes and the coordination game (and many others) have more than one NE.

BIG ISSUE.

- players may have different (opposite) preferences on the equilibrium outcomes (see BoS)
- it is not possible to speak of equilibrium strategies. In the BoS T is an equilibrium strategy? Or B?

One more problem

Example: matching pennies (MP)

$I \backslash II$	L	R
T	$-1 \quad 1$	$1 \quad -1$
B	$1 \quad -1$	$-1 \quad 1$

There is no equilibrium.

But Nash is famous (also) because of his existence thm (1950).
But MP is a zero-sum game. So, even vN (1928) guarantees that it has an equilibrium.
Where do we find it? Usual math trick: extend ($\mathbb{N}$ to $\mathbb{Z}$; sum to integral; solution to weak solution).

Mixed strategies

The basic idea is that the player does not choose a strategy, but a probability distribution on strategies.

Example: I have an indivisible object and I must assign it *in a fair way* to one of you. It is quite possible that the best solution is to **decide** to assign it **randomly** (with a uniform probability distribution).

Example: I am a DM and there are two optimal solutions. I will flip a coin, not willing to starve as Buridano' donkey. Or I should say *l'âne de Buridan*? Or maybe one should read Aristotle's *De Caelo*?

Mixed extension of a game

Let's apply it to games in strategic form.
Given a game $G = (X, Y, f, g)$, assume X, Y are finite, and let $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$.
The mixed extension of G is $\Gamma = (\Delta(X), \Delta(Y), \hat{f}, \hat{g})$.
Where:

$\Delta(X)$ is the set of probability distributions on X. An element of $\Delta(X)$ is $p = (p_1, \ldots, p_m) \in \mathbb{R}^m$. With...
$\hat{f}(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j f(x_i, y_j)$. Of course, $(p, q) \in \Delta(X) \times \Delta(Y)$.

Notice that Γ is itself a game in strategic form. So, no need to redefine concepts (in particular, NE).

Interpretation?

Of course, there is no mathematical problem in the definition of Γ.

But: $\hat{f}$ and $\hat{g}$ can still interpreted as payoffs for the players?

The answer is YES if the original u and v are vNM utility functions. Otherwise, we cannot attach a meaning to the operations that brought us form G to Γ.

Nash existence theorem

Theorem:
Given a game (X, Y, f, g) s.t.:
- X, Y are (not empty) convex, closed and bounded subsets of finite-dimensional Euclidean spaces
- f, g are continuous (on $X \times Y$)
- $f(\cdot, y)$ and $g(x, \cdot)$ are concave (quasi concave is enough)

Note: the mixed extension of a finite game satisfies these assumptions.

Proof

Idea of best-reply (correspondence).
$R_I(y) = \text{argmax}_{x \in X}\{f(x,y)\}$
Similarly for R_{II}.

Remark: are correspondences (multi-valued, possibly empty valued: make examples yourselves).

From R_I and R_{II} we build $R : X \times Y \rightrightarrows X \times Y$, defined as
$R(x,y) = R_I(y) \times R_{II}(x)$.
R satisfies the assumptions of Kakutani's thm. So, R has a "fixed point", i.e. there is $(\bar{x},\bar{y}) \in R(\bar{x},\bar{y})$.

Lemma: $(\bar{x},\bar{y}) \in R(\bar{x},\bar{y})$ iff $(\bar{x},\bar{y})$ is NE (NB: no assumption is needed).

Mixed extension and equilibria for BoS

The BoS is:

$I \backslash II$	L	R
T	2 1	0 0
B	0 0	1 2

Instead of using $((p_1, p_2), (q_1, q_2))$, we use: $((p, 1-p), (q, 1-q))$, with $p, q \in [0, 1]$. So:
- $\hat{f}((p, 1-p), (q, 1-q)) = 2pq + 1(1-p)(1-q)$.

Given $\bar{q}$, the best reply for player I to $\bar{q}$, is (easily seen from: $\hat{f}((p, 1-p), (\bar{q}, 1-\bar{q})) = (3\bar{q}-1)p + (1-\bar{q})$):

- $\{0\}$ for $0 \leq \bar{q} < 1/3$
- $[0, 1]$ for $\bar{q} = 1/3$
- $\{1\}$ for $1/3 < \bar{q} \leq 1$

Mixed extension and equilibria for BoS

The best reply for II is:

- $\{0\}$ for $0 \leq \bar{p} < 2/3$
- $[0,1]$ for $\bar{p} = 2/3$
- $\{1\}$ for $2/3 < \bar{p} \leq 1$

We can draw the picture that is in the following slide.

Notice that it can be proved that:

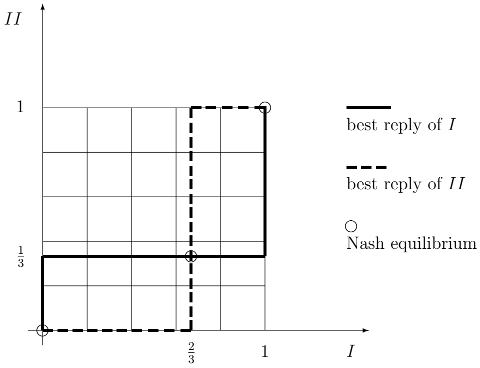
$$(\bar{x}, \bar{y}) \in R(\bar{x}, \bar{y}) \Leftrightarrow \begin{cases} (\bar{x}, \bar{y}) \in \mathrm{gph}(R_{II}) \\ (\bar{y}, \bar{x}) \in \mathrm{gph}(R_{I}) \end{cases}$$

or that:

$$(\bar{x}, \bar{y}) \in R(\bar{x}, \bar{y}) \Leftrightarrow (\bar{x}, \bar{y}) \in (\mathrm{gph}(R_{II})) \cap (\mathrm{gph}(R_{I}^{-1}))$$

So, we see from the picture that there are 3 NE

BoS: graphs of the best reply correspondences



Mixed extension of BoS: some comments

- Notice that the two NE that we had for the original game are preserved in the mixed extension. This is a general fact, valid for all games in strategic form. The original equilibria are to be found as equilibria in the mixed extension having all of their coordinates represented by a Dirac probability measure (probability concentrated on a single element of X, et cetera).

- The new, mixed equilibrium could be seen as a reasonable compromise, in the sense that each of the two players is precisely in the same situation compared with the two pure NE. The problem is that the compromise is inefficient. The payoff in the mixed equilibrium is $2/3$ for each player.

Mixed extension of BoS: some comments

Exercise: is it correct to say that player I prefers the NE (T, R) to (B, L) because in the first one has an utility value higher that player II? If not, why he prefers the NE (T, R) to (B, L)?

Exercise: why can we say that players are "equally maltreated" in the mixed equilibrium?

Exercise: prove that NE for a finite game G do not disappear passing to Γ.

BoS and correlated equilibria

- Why this inefficiency? Because, due to the fact that the players use different (independent) random devices, a lot of probability is "lost" in the sense that it is assigned to cells that contain $(0, 0)$ ($5/9$ is the probability of getting the payoffs $(0, 0)$).

- Any way to avoid this? Yes, using *the same random device* for both players.

- Consider the following (non binding) agreement: "It is tossed a fair coin, If head shows, player I will play T and player II will play R; if tails show, I plays B and II plays L".

This agreement is quite stable. It is an example of correlated equilibrium (Aumann, 1974).